Introduction to Proxy Vector Autoregressions

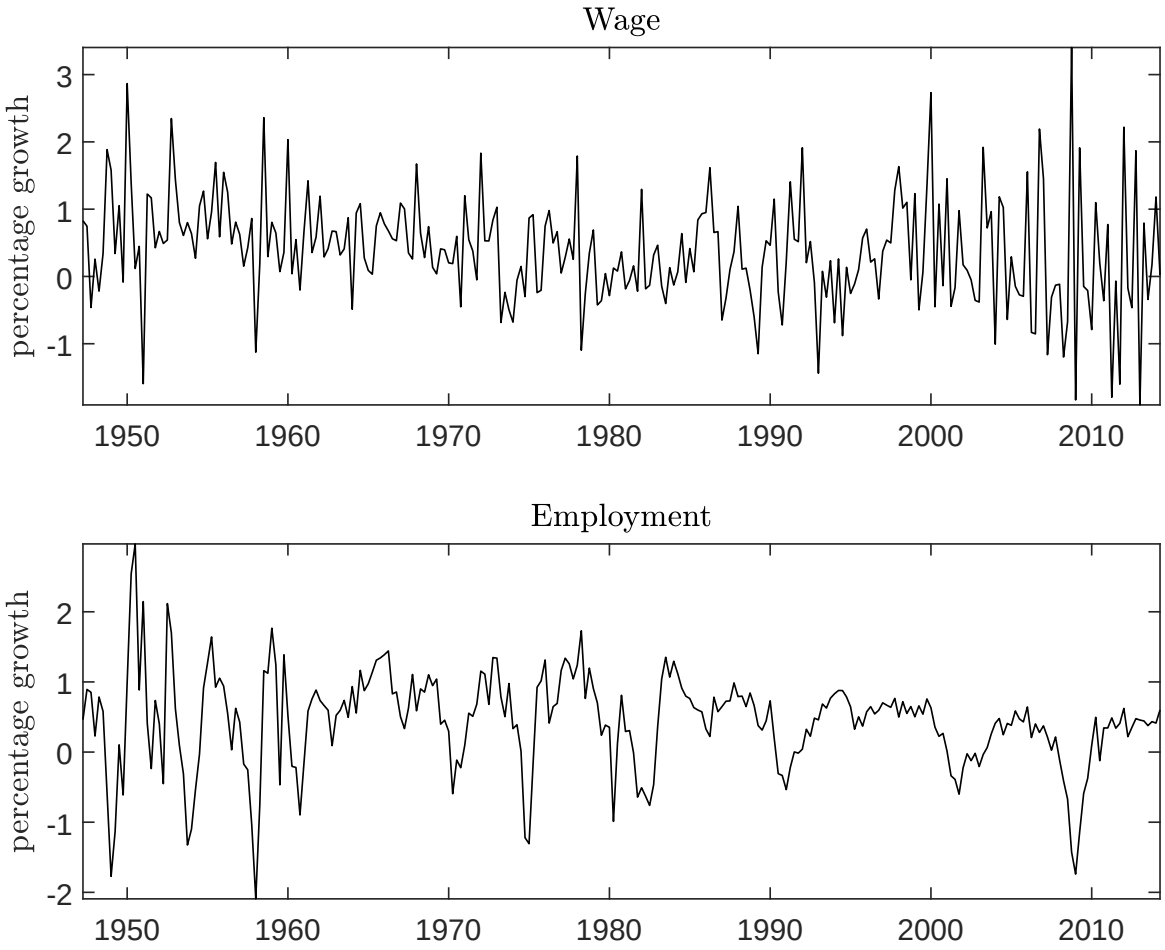
Martin Bruns

University of St Andrews

Introduction

- We have seen vector autoregressive models and their identification via zero restrictions, sign restrictions, or a combination of sign and zero restrictions
- Reminder: Every model variable depends on its own as well as lags of every other model variable
- We saw the VAR(p): VAR model with p autoregressive lags
- Labour market example (without intercept):

$$\begin{aligned}\Delta w_t &= a_{11,1}\Delta w_{t-1} + a_{12,1}\Delta empl_{t-1} \\ &\quad + \cdots + a_{11,p}\Delta w_{t-p} + a_{12,p}\Delta empl_{t-p} + u_{1t} \\ \Delta empl_t &= a_{21,1}\Delta w_{t-1} + a_{22,1}\Delta empl_{t-1} \\ &\quad + \cdots + a_{21,p}\Delta w_{t-p} + a_{22,p}\Delta empl_{t-p} + u_{2t} \end{aligned} \tag{1}$$


Wage
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1. Structural Vector Autoregressions

2. SVAR Tools
   - Structural Impulse Response Functions
   - Forecast Error Variance Decomposition
   - Historical Decomposition

3. Proxy Identification

4. Conclusion

Introduction

- We saw how to write the model in more compact form:

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t, \tag{2}$$

where

- u_t serially uncorrelated error term (if lag length is chosen appropriately)
- $Var(u_t) = \Sigma$
- We distinguished between the reduced form and the structural form of the model
- Reduced form in (2) proved useful
  - in summarizing the properties of many macroeconomic data series
  - for forecasting
  - for testing existence of equilibrium relationships

Introduction

Structural Form

- What are the dynamic causal effects of...
  - An exogenous increase in the policy rate?
  - An oil supply shock?
  - A labour demand shock?
- How does the Central Bank react to deviations of inflation/output from their targets?
- What is the fiscal policy rule?

Structural form

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + B w_t$$
$$w_t \sim N(\mathbf{0}, I)$$

- Effects of structural shocks w_t
  - Mutually uncorrelated (diagonal covariance matrix)
  - Distinct economic interpretation
  - Not directly observable
  - Allows statements about **causal** effects of variations in w_t on the data y_t

- Key step: Finding (some columns of) B
- Allows to construct
  1. Structural impulse responses
  2. Forecast error variance decomposition
  3. Historical decomposition

References and Further Codes

References

- Notation partly taken from Kilian & Lütkepohl (2017)
- Data and labour market example from Baumeister & Hamilton (2015)

Further Codes

- BEAR toolbox
- Andrew Blake and Haroon Mumtaz
- Gary Koop
- Lutz Kilian and Helmut Lütkepohl

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SVAR tools

SVAR model allows to construct statistics that answer the following questions:

1. What are the effects over time of an identified shock of a given size on the model variables? $\rightarrow$ structural impulse responses (structural IRF)
2. How much of the forecast error variance (or prediction mean squared error) of the endogenous variables at a certain horizon is accounted for by each structural shock? $\rightarrow$ forecast error variance decomposition (FEVD)
3. How much does a given structural shock explain of the historically observed fluctuations in the VAR variables? $\rightarrow$ historical decomposition (HD)

Motivation

- The main interest is not in the structural shocks w_t themselves
- Rather, in the reaction of the variables to these shocks
- Formally

$$\frac{\partial y_{t+h}}{\partial w_t'} = \Theta_h, \quad h = 0, \cdots, H \tag{3}$$

- Assuming we have an estimate for B, the structural IRFs can be computed from the slope coefficients $A_1, \cdots, A_p$

Motivation

- Step 1: Reduced form IRFs (reaction to a forecast error u_t): $\Phi_h = \sum_{j=1}^{h} \Phi_{h-j} A_j$
- Structural IRFs: $\Theta_h = \Phi_h B$
- Properties in linear VAR models:
  - Reaction to a positive shock is the mirror image of the reaction to a negative shock
  - Rescaling the size of the shock rescales the impulse response one-to-one

Normalization

- Structural IRFs at horizon $h = 0$ (impact effect) need to be normalised
- The reason is that from the data we can identify u_t
- But we can rewrite $u_t = Bw_t = BMM^{-1}w_t$, where M is diagonal
- Even if B is point-identified

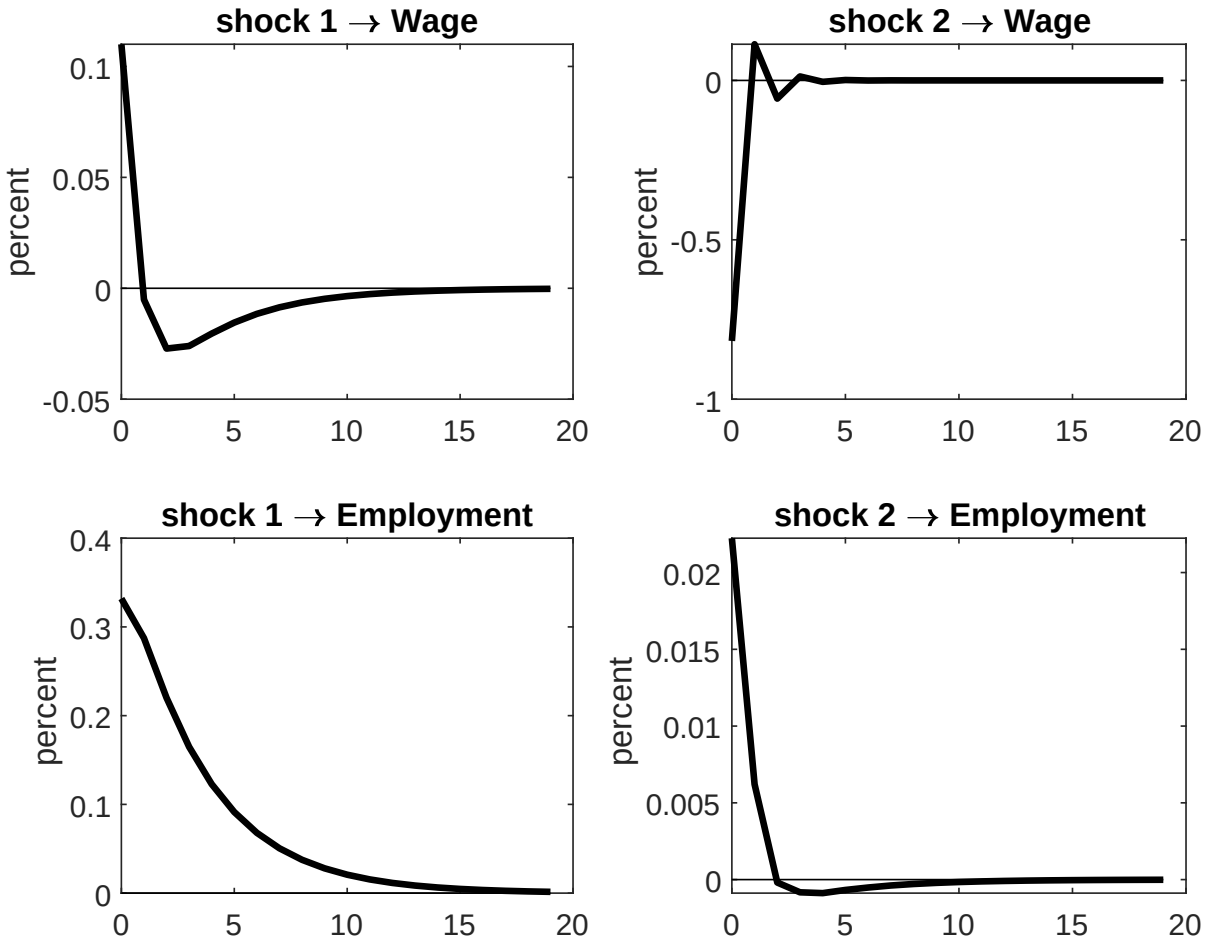
Normalization

- Three popular approaches to normalisation:

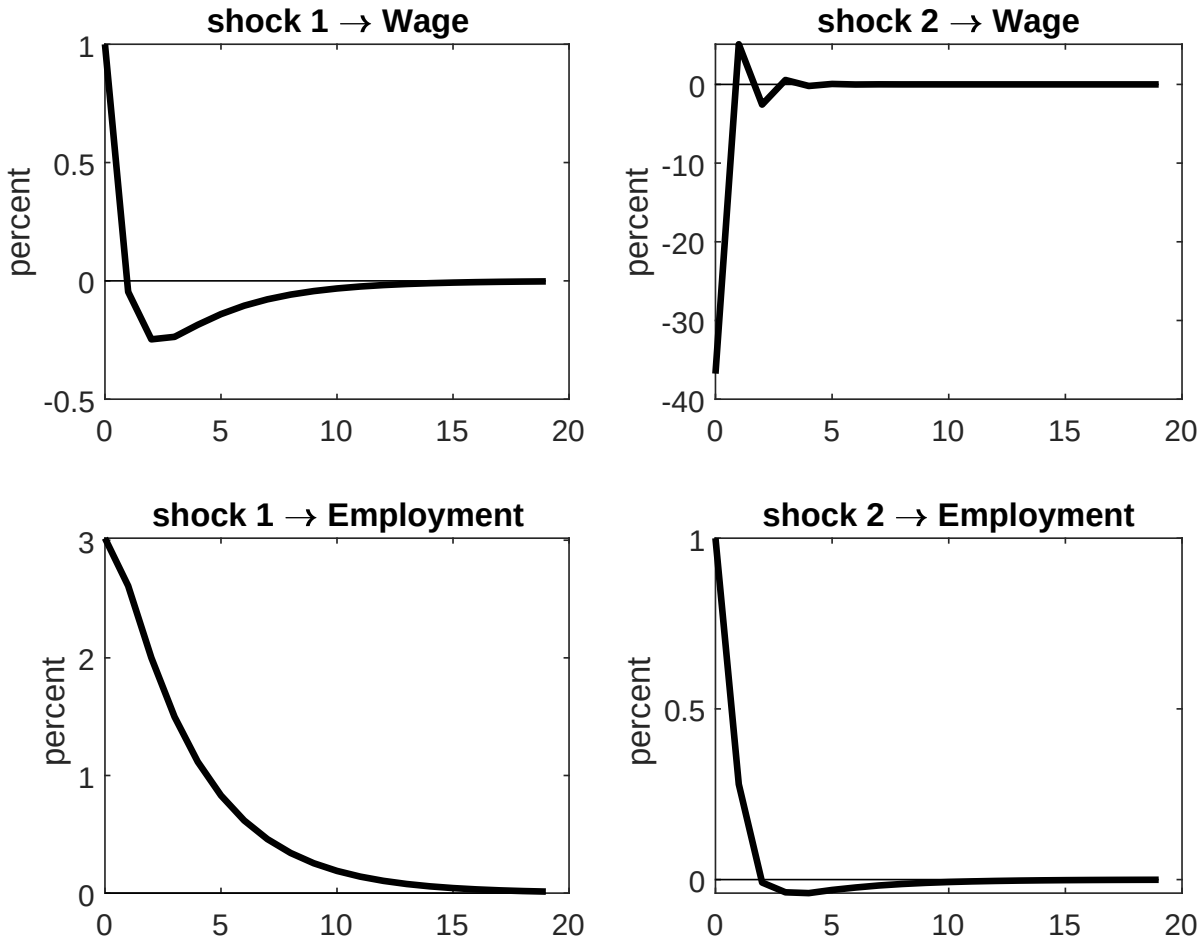
1. One standard deviation normalisation: provides the effect of a structural shock of size one standard deviation
2. Unit-effect normalisation: Provides the effect of a structural that moves one of the variables by 1
3. Other effect-normalisations: Often, IRFs to monetary policy shocks are normalised such that on impact the policy rate moves by 25bp

SVARtools.m, Section3

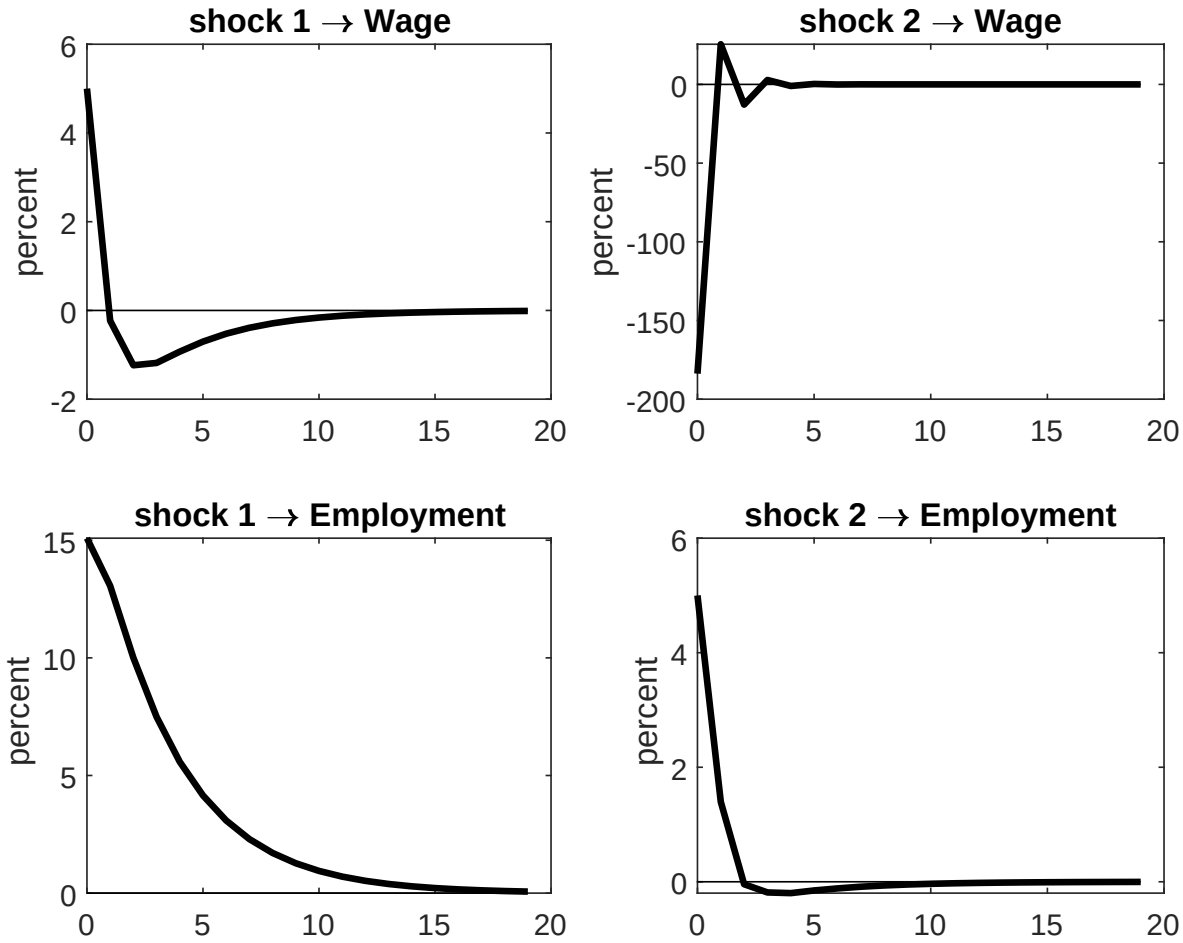
Standard Deviation Normalisation



Unit Effect Normalisation



Other Effect Normalisation



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FEVD: Motivation

- We've seen structural IRFs: What is the response of the model variables to an identified structural shock over time?
- Another popular question: How much of the forecast error variance at each horizon is accounted for by each structural shock? $\rightarrow$ FEVD
- FEVD can be computed directly from the structural IRF parameters $\Theta_0, \cdots, \Theta_H$

Computation

- h -step ahead forecast error: $FE(h) = y_{t+h} - y_{t+h|t}$
- Prediction mean squared error:
  $MSPE(h) = E(FE(h)^2) = \sum_{i=0}^{h-1} \Theta_i \Theta_i'$
- Contribution of shocks j to the MSPE of y_{kt} at horizon h:
  $MSPE_j^k(h) = \theta_{kj,0}^2 + \cdots + \theta_{kj,h-1}^2$
- Total MSPE: $MSPE^k(h) = \sum_{j=1}^{K} MSPE_j^k(h)$

  $1 = \frac{MSPE_1^k(h)}{MSPE^k(h)} + \cdots + \frac{MSPE_K^k(h)}{MSPE^k(h)}$
- Forecast Error Variance Decomposition (in percent):

  $FEVD_j^k(h) = 100 * \frac{MSPE_j^k(h)}{MSPE^k(h)}$

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Forecast Error Variance Decomposition



HD: Motivation

- For now, we have seen structural IRFs and FEVD
- Both describe average moments of the data
  - IRF: Average reaction over the sample of the model variables to structural shocks
  - FEVD Average contribution over the sample of structural shocks to the forecast error variance decomposition

HD: Motivation

- Alternatively, one might be interested in the contribution of structural shocks to historically observed fluctuations in the data
- For example: Sdid monetary policy cause the 1982 recession?
- To answer this type of questions, we need historical decompositions (HD)

Computation

- Data approximation based on MA representation:

$$\hat{y}_t = \sum_{s=0}^{t-1} \Theta_s w_{t-s} \tag{4}$$

- Cumulative effect of each shock on the second variable:
- Effect of shock 1: $\hat{y}_{2t}^{(1)} = \sum_{i=0}^{t-1} \theta_{21,i} w_{1,t-i}$
- Effect of shock 2: $\hat{y}_{2t}^{(2)} = \sum_{i=0}^{t-1} \theta_{22,i} w_{2,t-i}$

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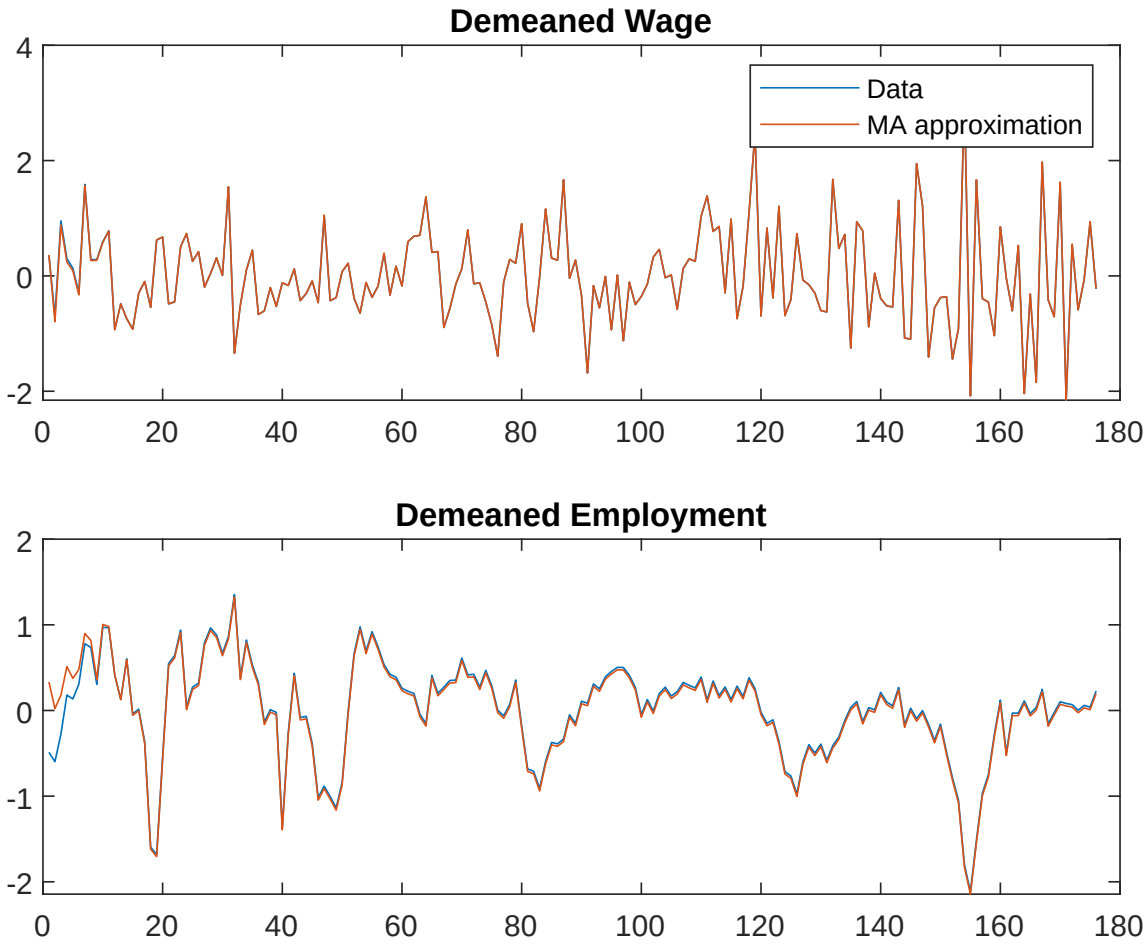
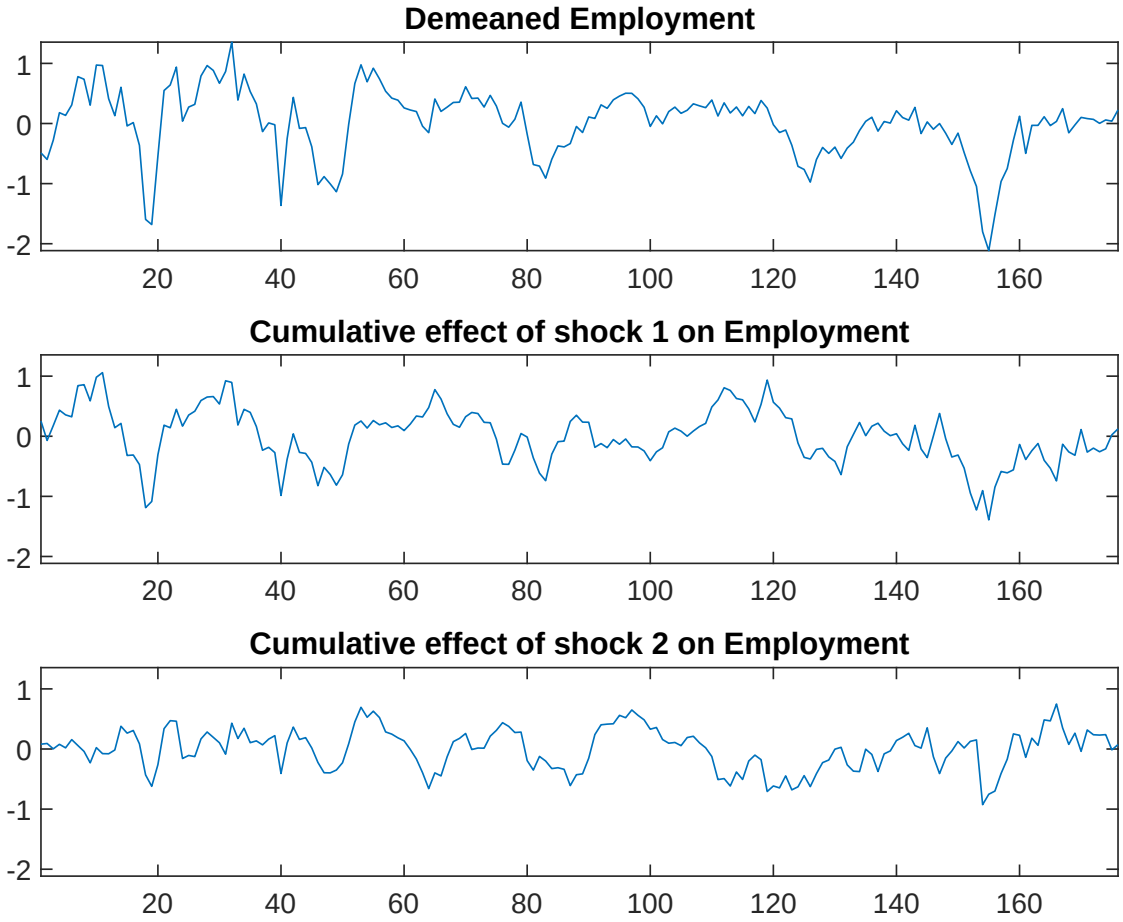
Moving Average Representation



Historical Decomposition



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Identification

- So far we have assumed that B is known
- The covariance restriction implies

$$BB' = \Sigma \tag{5}$$

Observational equivalence

- K^2 elements in B (K: number of variables)
- $\frac{K(K+1)}{2}$ distinct elements in Σ
- $\frac{K(K-1)}{2}$ restrictions missing for full and point identification

$$\{B\} \text{ s.t. } BB' = \Sigma \tag{6}$$

Proxy Identification

- One way to achieve identification is the use of proxies, z_t
- Sometimes thought of as the shocks w_t observed with measurement error.
- some studies assume

$$z_t = \phi w_t + \eta_t, \tag{7}$$

- where ϕ is a scalar and η_t is a measurement error
- e.g. Mertens & Ravn (2013), Jentsch & Lunsford (2019), Caldara & Herbst (2019), Bruns & Lütkepohl (2023)

Proxy Identification

- Other studies assume a richer proxy data generating process
- These can include for example lags of the proxies or the endogenous variables
- e.g. Stock & Watson (2018) Plagborg-Møller & Wolf (2021)

Proxy Identification

- Assume there exists a set of N instrumental variables (proxies) $z_t = (z_{1t}, \ldots, z_{Nt})'$

$$\mathbb{E}(\mathbf{w}_{1t} z_t') = \Sigma_{\mathbf{w}_1 z} \neq 0, \quad \Sigma_{\mathbf{w}_1 z} \ (K_1 \times N), \quad \text{rk}(\Sigma_{\mathbf{w}_1 z}) = K_1 \quad \text{(relevance)}, \tag{8}$$

$$\mathbb{E}(\mathbf{w}_{2t} z_t') = 0 \quad \text{(exogeneity)}. \tag{9}$$

- Sometimes it is also required that

$$\mathbb{E}(\mathbf{w}_{t-j} z_t') = 0 \quad \forall j \neq 0 \quad \text{(lead-lag exogeneity)} \tag{10}$$

- Then we will see that z_t can potentially be used to identify K_1 columns of B

Proxy Identification

- The relevance and exogeneity conditions (8)/(9) imply

$$\mathbb{E}(u_t z_t') = B\mathbb{E}(\mathbf{w}_t z_t') = B_1 \Sigma_{\mathbf{w}_1 z}. \tag{11}$$

- In other words, the parameters, B_1, needed to identify the shocks of interest, have to satisfy the additional relations (11)
- If there is a scalar proxy ($N = 1$), B_1 is just a column vector and $\Sigma_{\mathbf{w}_1 z}$ is a scalar
- $\mathbb{E}(u_t z_t)$ is a scalar multiple of the impact effects of the shock of interest

Proxy Identification

- However: $\Sigma_{\mathbf{w}_1 z}$ is unknown
- Without further restrictions, an estimator based on $\mathbb{E}(u_t z_t')$ identifies a group of K_1 shocks jointly
- To disentangle the K_1 shocks of interest we need further restrictions
- If there are exactly as many proxies as there are shocks of interest ($N = K_1$) and each proxy individually satisfies the relevance and exogeneity conditions
- then $\Sigma_{\mathbf{w}_1 z}$ is a diagonal matrix that even over-identifies the shocks individually (see Bruns et al. 2025)

Proxy Identification

- We can place restrictions on
- B_1:
  1. exclusion or other just-identifying restrictions: e.g., Mertens & Ravn (2013), Känzig (2021)
  2. sign restrictions: e.g., Jarociński & Karadi (2020), Giacomini et al. (2022), and Braun & Brüggemann (2023)
  3. heteroskedasticity: Carriero et al. (2024) and Angelini et al. (2024))

Proxy Identification

- We can also place restrictions on $\Sigma_{\mathbf{w}_1 z}$:
  1. just-identifying: e.g., Lunsford (2015) and Lakdawala (2019)
  2. over-identifying: e.g., Bertsche & Braun (2022), Schlaak et al. (2023), Bruns et al. (2025)
  3. set-identifying: e.g. Piffer & Podstawski (2018), Ludvigson et al. (2021) and Arias et al. (2021)

Proxy Identification: Takeaways

- For a single structural shock $\sum_{t=1}^{T} u_t z_t$ identifies the first column of B up to scale
- For multiple proxies identifying K_1 structural shocks $\sum_{t=1}^{T} u_t z_t$ identifies a linear combination of B_1, the first K_1 columns of B
- To disentangle the K_1 shocks further identifying restrictions are necessary
- These restrictions can be placed on B_1 or on $\mathbb{E}(\mathbf{w}_t z_t') = \Sigma_{\mathbf{w}_1 z}$

Conclusions

- We distinguish between the reduced form and the structural form of a VAR model
- Transitioning from the reduced to the structural form requires **identification**
- Once the model is identified we can compute IRFs, FEVDs, and HDs
- We saw how proxies can be used for identification

Angelini, G., Fanelli, L. & Neri, L. (2024), 'Invalid proxies and volatility changes', *arXiv preprint arXiv:2403.08753* .

Arias, J. E., Rubio-Ramírez, J. F. & Waggoner, D. F. (2021), 'Inference in Bayesian Proxy-SVARs', *Journal of Econometrics* **225**, 88–106.

Baumeister, C. & Hamilton, J. D. (2015), 'Sign restrictions, structural vector autoregressions, and useful prior information', *Econometrica* **83**(5), 1963–1999.

Bertsche, D. & Braun, R. (2022), 'Identification of structural vector autoregressions by stochastic volatility', *Journal of Business & Economic Statistics* **40**(1), 328–341.

Braun, R. & Brüggemann, R. (2023), 'Identification of SVAR models by combining sign restrictions with external instruments', *Journal of Business & Economic Statistics* **41**(4), 1077–1089.

Bruns, M. & Lütkepohl, H. (2023), 'An alternative bootstrap for proxy vector autoregressions', *Computational Economics* **62**, 1857–1882.

Bruns, M., Lütkepohl, H. & McNeil, J. (2025), 'Avoiding unintentionally correlated shocks in proxy vector autoregressive analysis', *Journal of Business and Economic Statistics* . forthcoming.

Caldara, D. & Herbst, E. (2019), 'Monetary policy, real activity, and credit spreads: Evidence from Bayesian proxy SVARs', *American Economic Journal: Macroeconomics* **11**, 157–192.

Carriero, A., Marcellino, M. & Tornese, T. (2024), 'Blended identification in structural VARs', *Journal of Monetary Economics* **146**, 103581.

Giacomini, R., Kitagawa, T. & Read, M. (2022), 'Robust Bayesian inference in proxy SVARs', *Journal of Econometrics* **228**(1), 107–126.

Jarociński, M. & Karadi, P. (2020), 'Deconstructing monetary policy surprises - the role of information shocks', *American Economic Journal: Macroeconomics* **12**, 1–43.

Jentsch, C. & Lunsford, K. G. (2019), 'The dynamic effects of personal and corporate income tax changes in the United States: Comment', *American Economic Review* **109**, 2655–2678.

Känzig, D. R. (2021), 'The macroeconomic effects of oil supply news: Evidence from OPEC announcements', *American Economic Review* **111**(4), 1092–1125.

Kilian, L. & Lütkepohl, H. (2017), *Structural Vector Autoregressive Analysis*, Cambridge University Press, Cambridge.

Lakdawala, A. (2019), 'Decomposing the effects of monetary policy using an external instruments SVAR', *Journal of Applied Econometrics* **34**, 934–950.

Ludvigson, S. C., Ma, S. & Ng, S. (2021), 'Uncertainty and business cycles: exogenous impulse or endogenous response?', *American Economic Journal: Macroeconomics* **13**(4), 369–410.

Lunsford, K. (2015), Identifying structural VARs with a proxy variable and a test for a weak proxy, Technical report, Federal Reserve Bank of Cleveland.

Mertens, K. & Ravn, M. O. (2013), 'The dynamic effects of personal and corporate income tax changes in the United States', *American Economic Review* **103**, 1212–1247.

Piffer, M. & Podstawski, M. (2018), 'Identifying uncertainty shocks using the price of gold', *The Economic Journal* **128**(1549), 3266–3284.

Plagborg-Møller, M. & Wolf, C. K. (2021), 'Local projections and VARs estimate the same impulse responses', *Econometrica* **89**, 955 – 980.

Schlaak, T., Rieth, M. & Podstawski, M. (2023), 'Monetary policy, external instruments, and heteroskedasticity', *Quantitative Economics* **14**(1), 161–200.

Stock, J. H. & Watson, M. W. (2018), 'Identification and estimation of dynamic causal effects in macroeconomics using external instruments', *Economic Journal* **128**, 917–948.