Proxy Vector Autoregressions and Proxy Local Projections

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Introduction

- Macroeconomists often seek to analyse impulse response functions (IRFs):

$$\frac{\partial y_{t+h}}{\partial w_t'} = \Theta_h, \quad h = 0, \cdots, H \tag{1}$$

- where
  - w_t: structural shock
  - y_{t+h}: outcome variable
  - H: impulse resposne horizon
- Two popular approaches exist:
  1. Vector autoregressions (VARs): Sims (1980)
  2. Local Projections (LPs): Jordà (2005)

Introduction

- VAR-based IRFs iterate on a flexible dynamic multivariate model
- They use the autoregressive slope coefficients of the VAR
- This is analogous to iterative forecasting
- LP-based IRFs directly regress a future outcome y_{t+h} on current covariates
- Separate regressions are run for each horizon h
- This is analogous to direct forecasting

1. Introduction
2. VAR and LP
3. Bias-variance trade-off
4. Comparison of Proxy Local Projection Estimators
   - Motivation
   - Finding 1
   - Finding 2
   - Finding 3
   - Finding 4
   - Conclusion
5. Summary

VAR and LP

- VAR model:

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t, \tag{2}$$

- Iterating forward h periods gives the LP representation:

$$y_{t+h} = \nu_h + A^{(h+1)} Y_{t-1} + v_{t+h}^{(h)}, \tag{3}$$

- where intercept vector ν_h depends on the integer h
- $Y'_{t-1} = (y'_{t-1}, \ldots, y'_{t-p})$: Kp-dimensional vector of lagged dependent variables
-

$$v_t^{(h)} = u_t + \Phi_1 u_{t-1} + \cdots + \Phi_h u_{t-h} \tag{4}$$

  is a weighted sum of $u_t, \ldots, u_{t-h}$
- $A^{(h)} = [A_1^{(h)}, \ldots, A_p^{(h)}]$: the first K rows of the companion matrix $\mathbf{A}^h$

Equivalence between LP and VAR

- Shock propagation between VAR and LP differs:
  - LP: uses autocorrelations up to horizon h.
  - VAR: extrapolates from first p autocovariances.
- If p is large, methods become equivalent.
- Even for modest p, they are similar for horizons $h \leq p$.
- see Montiel Olea et al. (2025) for an overview

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Bias-variance trade-off

- VAR and LP IRFs differ in terms of bias and variance
- Robustness of LP to dynamic misspecification comes at significant variance cost
- Li et al. (2024) conduct a large-scale simulation study analysing this trade-off
- Here we are restricting attention to an illustrative example

Bias-variance trade-off

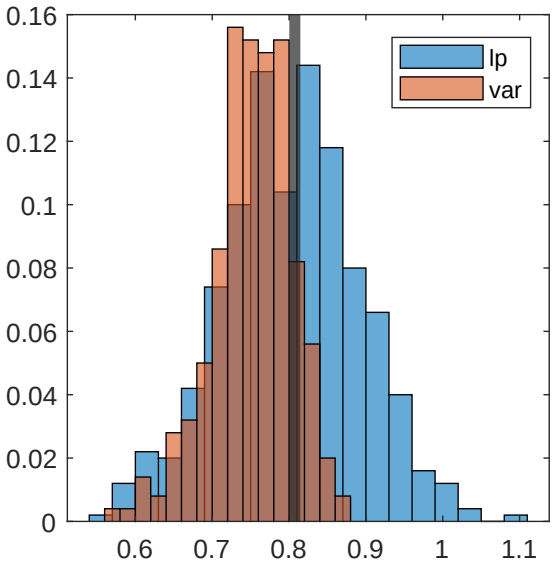
- If the true data generating process (DGP) is a finite-order VAR model then the VAR-based IRF estimator is always preferred
- Instead, while a low-order VAR is known to forecast well, it is unlikely to be literally true
- If the lag length is chosen too, low, moving average (MA) terms are introduced
- Other sources of MA terms are omitted variables, inappropriate data aggregation, and measurement errors

Bias-variance trade-off: Illustrative Example

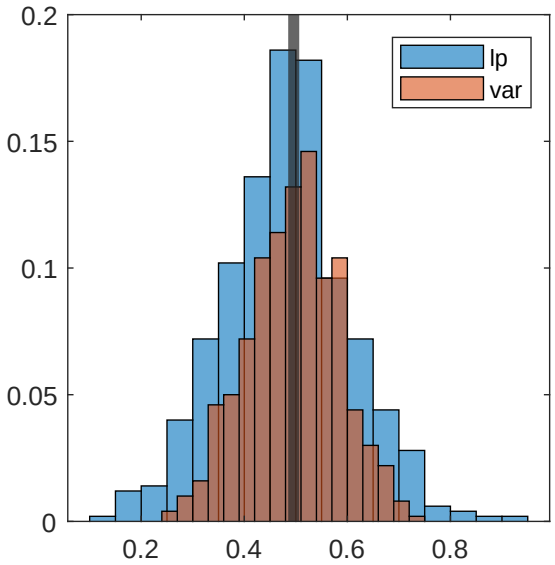
- We will consider as a DGP: $y_t = \rho y_{t-1} + w_t + \alpha w_{t-1}$ with $w_t \sim N(0,1)$
- Inspired by Plagborg-Møller (2026)
- Baseline: $\alpha = 0.1$, $\rho = 0.85$, $h = 2$, $T = 240$, $p = 1$
- 500 Simulations
- No bias correction
- True effect: $\rho^{h-1}(\alpha + \rho)$

Illustrative Example

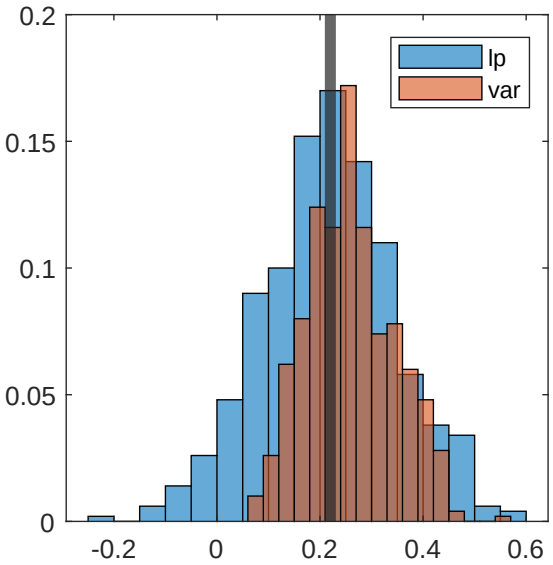
Effect of Impulse Response horizon (h)



$h = 2$



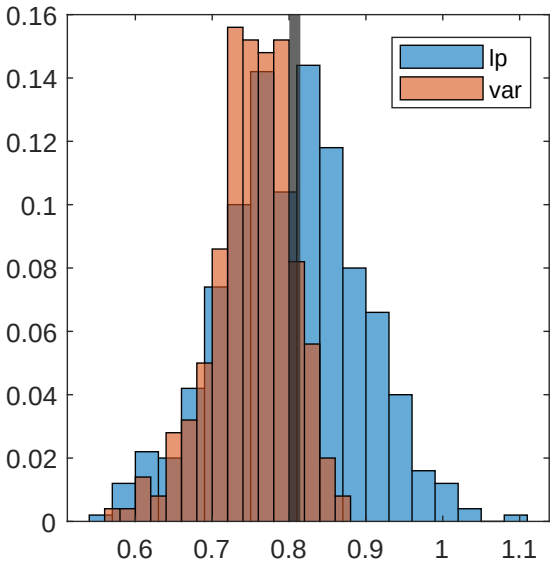
$h = 5$



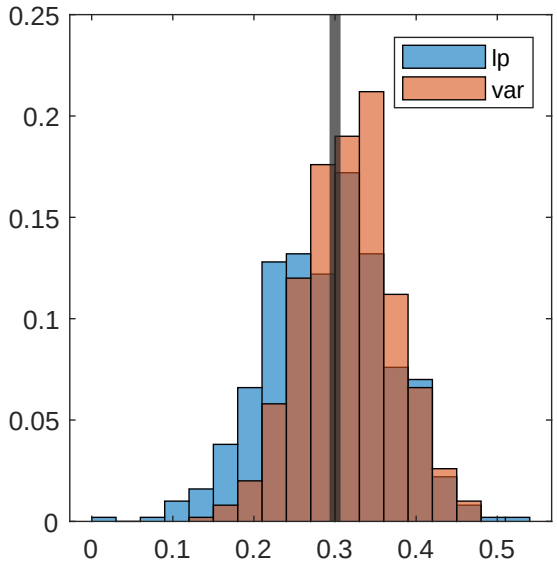
$h = 10$

Illustrative Example

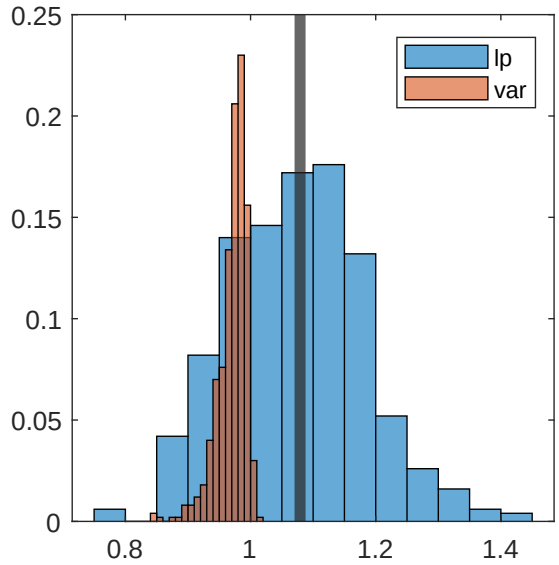
Effect of Persistence (ρ)



$\rho = 0.85$



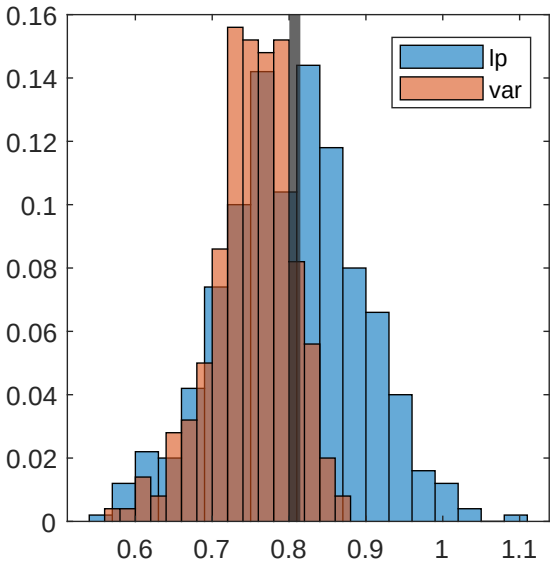
$\rho = 0.5$



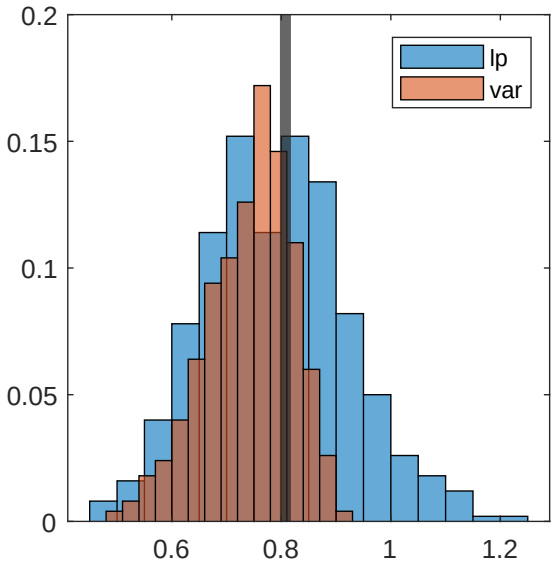
$\rho = 0.99$

Illustrative Example

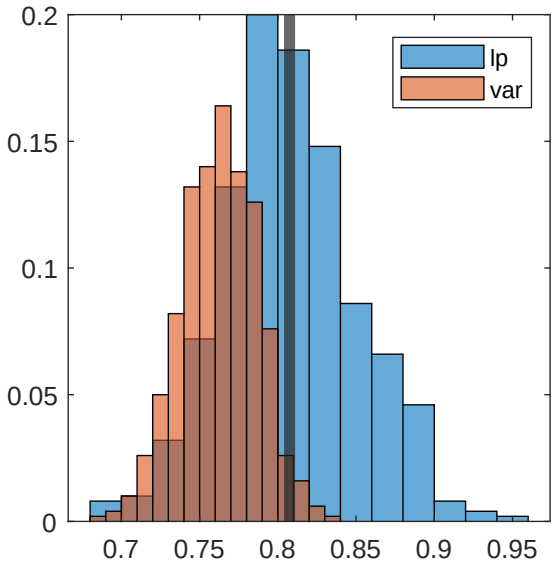
Effect of Sample Size (T)



$T = 240$



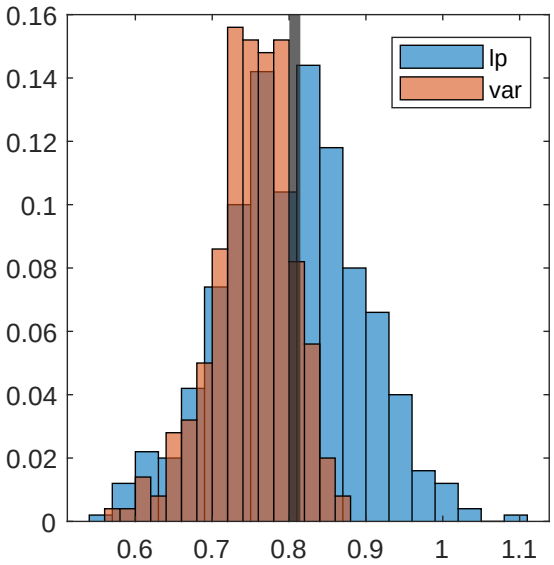
$T = 120$



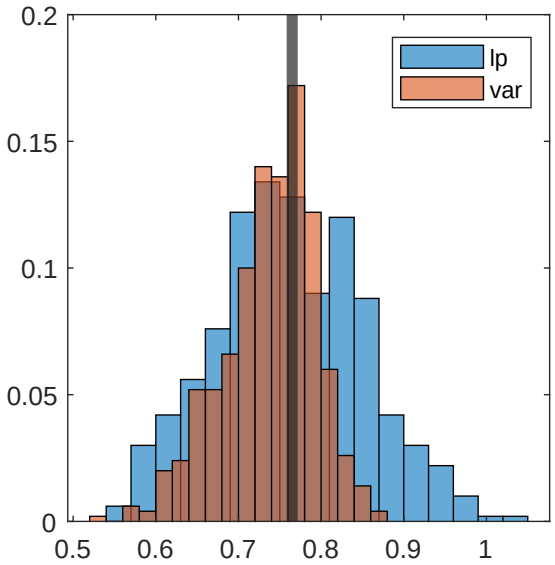
$T = 1000$

Illustrative Example

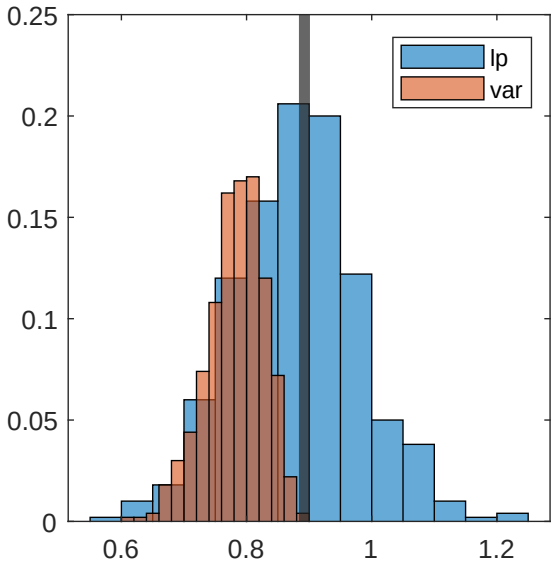
Effect of MA Contamination (α)



$\alpha = 0.1$



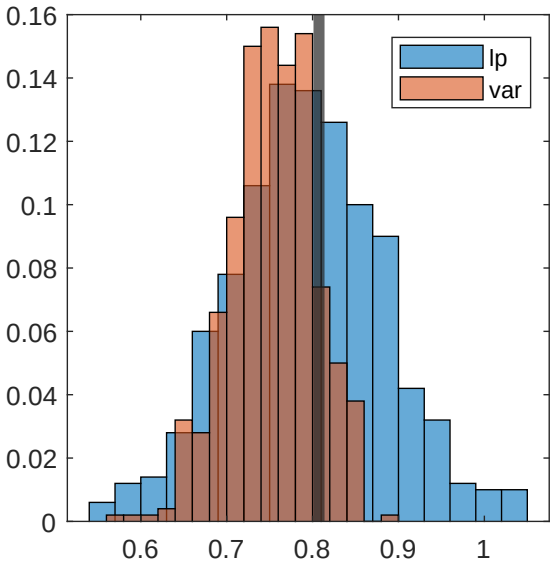
$\alpha = 0.05$



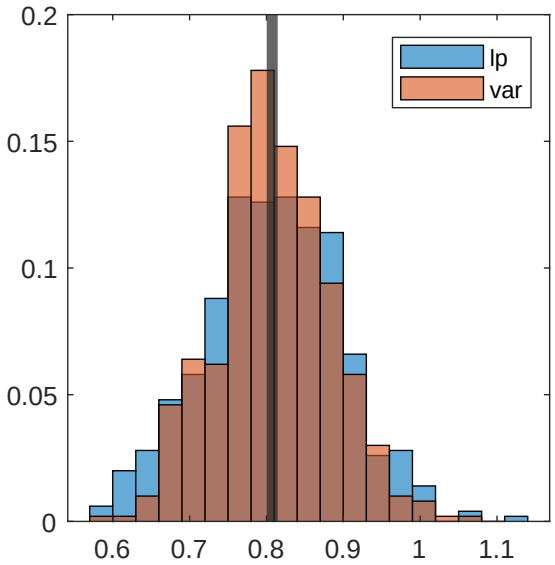
$\alpha = 0.2$

Illustrative Example

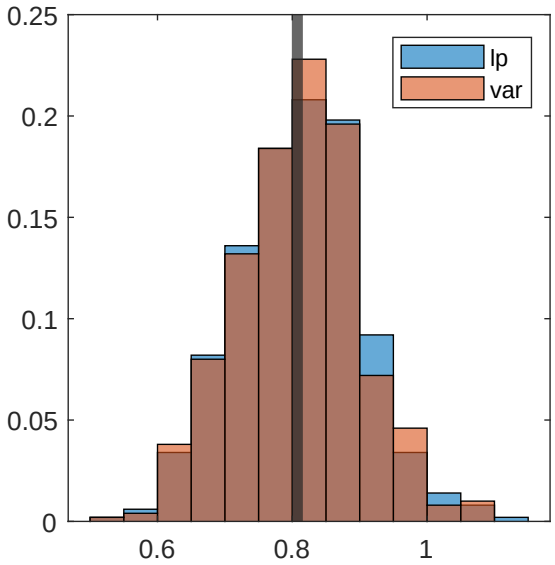
Effect of Lag Length (p)



$p = 1$



$p = 2$



$p = 3$

Illustrative Example: Take-aways

- Potentially stark differences in bias-variance trade-off in practice
- Robustness of LP to dynamic misspecification comes at significant variance cost
- So far we have considered the case where the structural shock, w_t, was known
- Next we will compare different LP estimators where identification is achieved via a proxy variable
- Simulation study based on Bruns & Lütkepohl (2022)

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Motivation

- Local Projection (LP) estimators increasingly popular alternative to vector autoregressions (VAR)
- In population: Equivalence between LP and VAR (Plagborg-Møller & Wolf 2021)
- In small sample: Trade-off between efficiency and robustness to misspecification
- Dependence on data-generating process (DGP) (Li et al. 2024)
- Battery of different LP estimators, suitable for various empirical circumstances
- This paper: Taking stock
  - Comparison of 10 LP estimators and VAR
  - Two new, residual-based LP type estimators
  - Similarities and numerical equivalences
  - Monte Carlo study: DGP is a finite order proxy VAR with invertible shocks

Main Findings

1. Numerical Equivalence between $\widehat{\Theta}_{LP}^{aug}$ and $\widehat{\Theta}_{BB}$
2. $\widehat{\Theta}_{LP}$ and $\widehat{\Theta}_{LP}^{aug}$ dominate $\widehat{\Theta}_{IV}$,$\widehat{\Theta}_{IV}^{y}$, and $\widehat{\Theta}_{IV}^{yz}$
3. $\widehat{\Theta}_{LP}^{resid}$ and $\widehat{\Theta}_{LP}^{ss}$ not competitive
4. $\widehat{\Theta}_{BB}^{GLS}$ dominates overall, $\widehat{\Theta}_{LP}^{GLS}$ is competitive

Estimators of structural impulse responses

Estimators included in comparison

1. $\widehat{\Theta}_{VAR}$: E.g. Kilian & Lütkepohl (2017) ▸ VAR
2. $\widehat{\Theta}_{LP}$: Jordà (2005) ▸ LP
3. $\widehat{\Theta}^{GLS}_{LP}$: Lusompa (2023) ▸ LP GLS
4. $\widehat{\Theta}^{aug}_{LP}$: Montiel Olea & Plagborg-Møller (2021) ▸ LP aug
5. $\widehat{\Theta}_{BB}$: Breitung & Brüggemann (2023) ▸ BB
6. $\widehat{\Theta}^{GLS}_{BB}$: Breitung & Brüggemann (2023) ▸ BB GLS
7. $\widehat{\Theta}_{IV}$: Stock & Watson (2018) ▸ IV
8. $\widehat{\Theta}^{y}_{IV}$: Stock & Watson (2018) ▸ IV^y
9. $\widehat{\Theta}^{yz}_{IV}$: Stock & Watson (2018) ▸ IV^{yz}
10. $\widehat{\Theta}^{resid}_{LP}$: our paper ▸ LP resid
11. $\widehat{\Theta}^{ss}_{LP}$: our paper ▸ LP ss

Finding 1: Numerical Equivalence between $\widehat{\Theta}_{LP}^{aug}$ and $\widehat{\Theta}_{BB}$

The General VAR Setup

"VAR representation"

$$y_t = \nu + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t, \quad u_t \sim (0, \Sigma_u) \tag{5}$$

"LP representation"

$$y_{t+h} = \nu_h + A^{(h+1)} Y_{t-1} + v_{t+h}^{(h)}, \tag{6}$$

where

- $Y_{t-1}' = (y_{t-1}', \ldots, y_{t-p}')$
- $v_t^{(h)} = u_t + \Phi_1 u_{t-1} + \cdots + \Phi_h u_{t-h}$
- reduced form IRFs

$$\Phi_i = \sum_{j=1}^{i} \Phi_{i-j} A_j \tag{7}$$

- $A^{(h)} = [A_1^{(h)}, \ldots, A_p^{(h)}]$ the first K rows of the hth power of the companion matrix

companion matrix

Finding 1: Numerical Equivalence between $\widehat{\Theta}_{LP}^{aug}$ and $\widehat{\Theta}_{BB}$

The General VAR Setup (cont.)

Identification

- $u_t = Bw_t$,
- $w_t \sim (0, D)$, D diagonal
- Structural IRFs: $\Theta_h = \Phi_h B$
- Three conditions for a proxy z_t:
  1. $\mathbb{E}(w_{1t} z_t) = c \neq 0$ (relevance)
  2. $\mathbb{E}(w_{kt} z_t) = 0, \quad k \neq 1$ (exogeneity)
  3. $\mathbb{E}(w_{t+i} z_t) = 0, \quad i \neq 0$ (lead-lag exogeneity)

$$\hat{\theta}_0 = \frac{\sum_{t=1}^{T} (\hat{u}_t z_t)}{\sum_{t=1}^{T} \hat{u}_{1t} z_t} \tag{8}$$

Finding 1: Numerical Equivalence between $\widehat{\Theta}_{LP}^{aug}$ and $\widehat{\Theta}_{BB}$

- Lag-augmented LP estimator (Montiel Olea & Plagborg-Møller 2021)

$$\hat{\Theta}(H)_{LP}^{aug} = [\hat{\theta}_0, \hat{\Phi}_1^{aug}\hat{\theta}_0, ..., \hat{\Phi}_H^{aug}\hat{\theta}_0], \tag{9}$$

  - $\hat{\theta}_0$ from equation (8)
  - $\hat{\Phi}_i^{aug} = \hat{A}_i^{(i)}$ from equation (6) (plus one lag)
- Projection estimator by Breitung & Brüggemann (2023)

$$\hat{\Theta}(H)_{BB} = [\hat{\theta}_0^{BB}, \hat{\theta}_1^{BB}, ..., \hat{\theta}_H^{BB}], \tag{10}$$

  - from OLS regression: $y_{t+h} = \nu_h + A^{(h+1)}Y_{t-1} + \theta_h \hat{w}_{1t} + e_t^{(h)}$
  - where $\hat{w}_{1t}$ obtained in a preliminary projection step:
    a. Estimate $(\hat{u}_{2t}, .., \hat{u}_{Kt}) = \gamma \hat{u}_{1t} + w_t^{(2)}$ via instrumental variables estimator
    b. Estimate $\hat{u}_{1t} = \eta \hat{w}_t^{(2)} + w_{1t}$

▸ Numerical Equivalences

Finding 2: $\widehat{\Theta}_{LP}$ and $\widehat{\Theta}^{aug}_{LP}$ dominate $\widehat{\Theta}_{IV}$, $\widehat{\Theta}^{y}_{IV}$, and $\widehat{\Theta}^{yz}_{IV}$

Monte Carlo: Data Generating Processes

DGP1

- Bivariate VAR(1): $y_t = A_1 y_{t-1} + u_t, \quad u_t = B w_t$
- $A_1 = \begin{bmatrix} a_{11} & 0 \\ 0.5 & 0.5 \end{bmatrix}$, $a_{11} = \{0.1, 0.5, 0.9, 0.95, 1\}$,
- $B = \begin{bmatrix} 1 & 0 \\ 0.5 & 3 \end{bmatrix}$
- Proxy: $z_t = \phi w_{1t} + \eta_t$, $\eta_t \sim N(0, \sigma^2_\eta)$

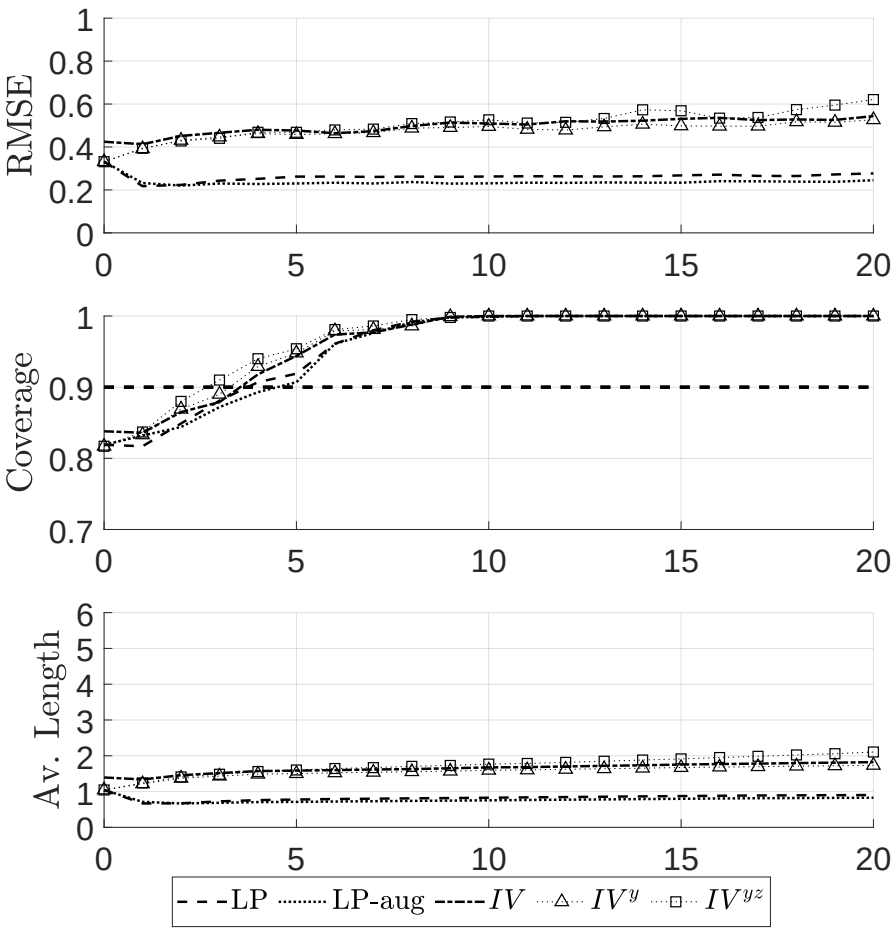
Case	ϕ	σ^2_η	$\text{Corr}(w_{1t}, z_t)$
strong proxy	1	0.2346	0.9 $\forall t$
weak proxy	1	3	0.5 $\forall t$

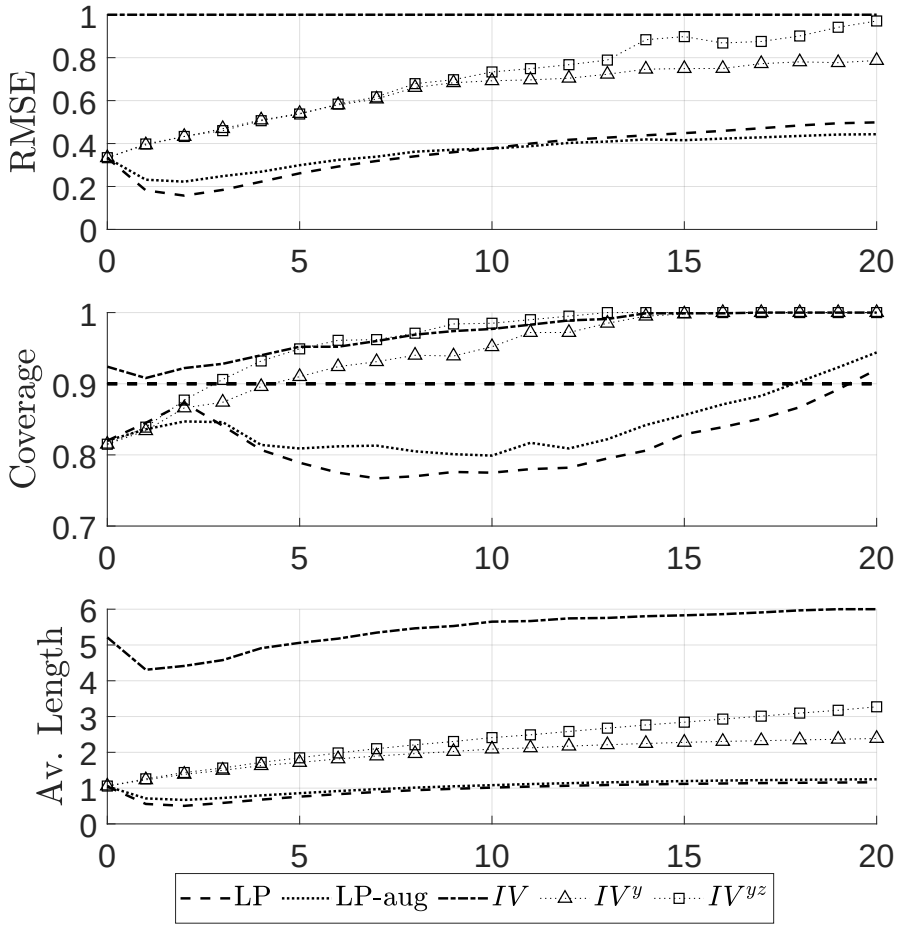
DGP2

- VAR(1) model based on 7 variables and proxy by Mertens & Ravn (2013)

Finding 2: $\widehat{\Theta}_{LP}$ and $\widehat{\Theta}^{aug}_{LP}$ dominate $\widehat{\Theta}_{IV}$, $\widehat{\Theta}^{y}_{IV}$, and $\widehat{\Theta}^{yz}_{IV}$

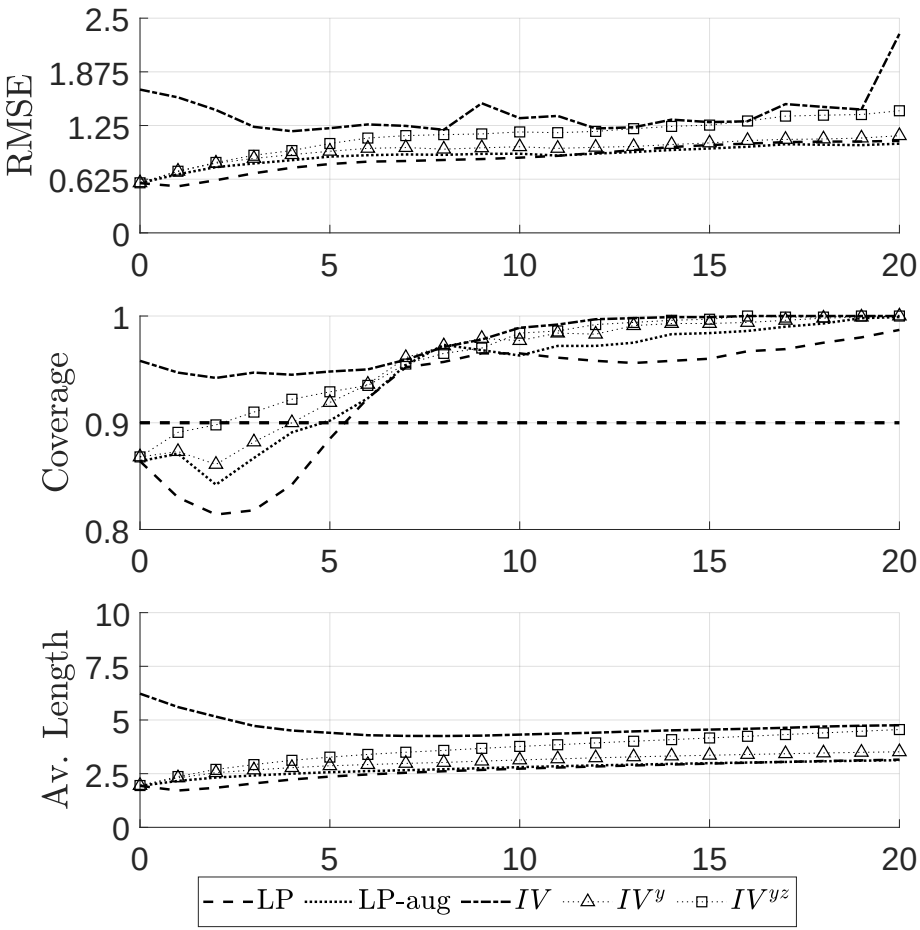
Pointwise MBB 90% confidence bands, $T = 100$, $p = 1$, with bias corr., $corr. = 0.9$, $a_{11} = 0.5$ (left), $a_{11} = 0.9$ (right), DGP1

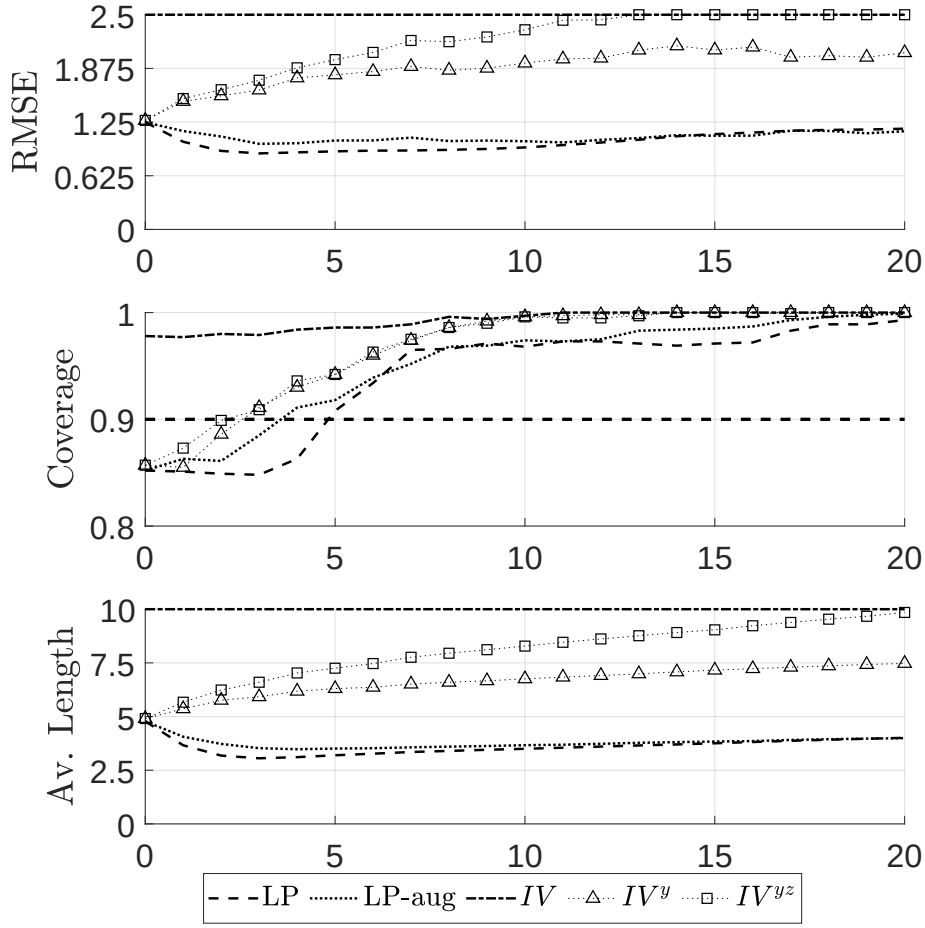




Finding 2: $\widehat{\Theta}_{LP}$ and $\widehat{\Theta}^{aug}_{LP}$ dominate $\widehat{\Theta}_{IV}$, $\widehat{\Theta}^{y}_{IV}$, and $\widehat{\Theta}^{yz}_{IV}$

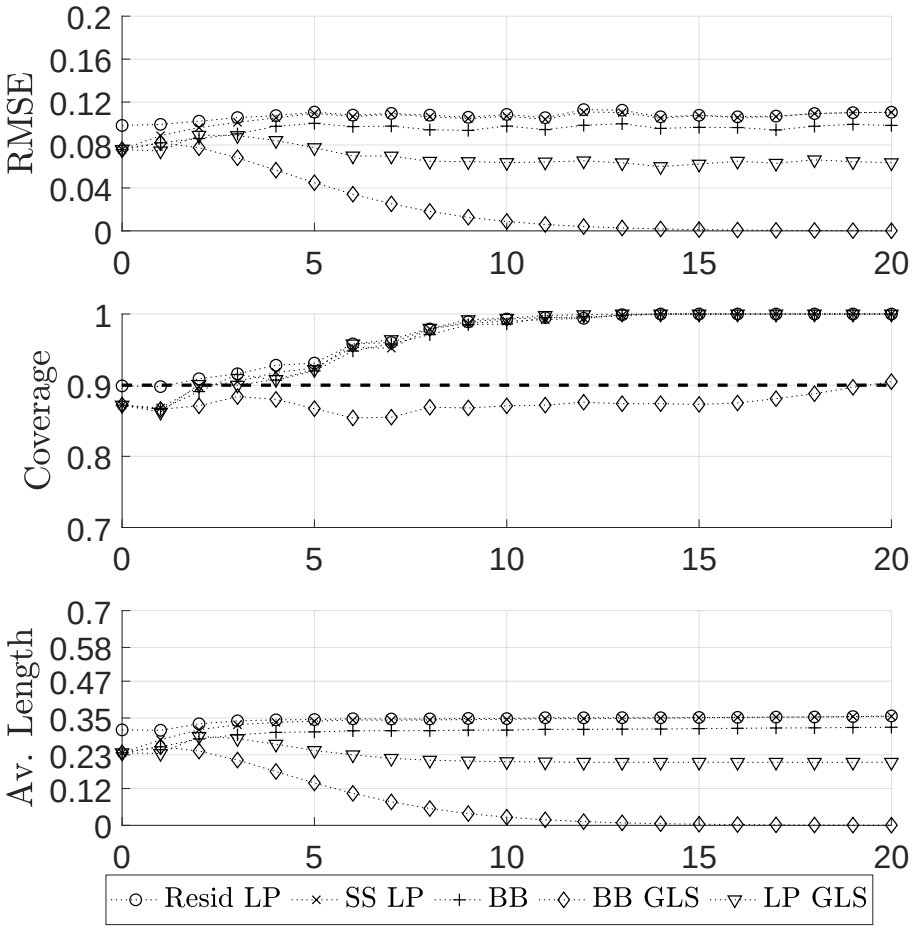
Pointwise MBB 90% confidence bands, $T = 200$, $p = 1$, with bias corr.,
corr. $= 0.9$ (left), *corr.* $= 0.5$ (right), DGP2

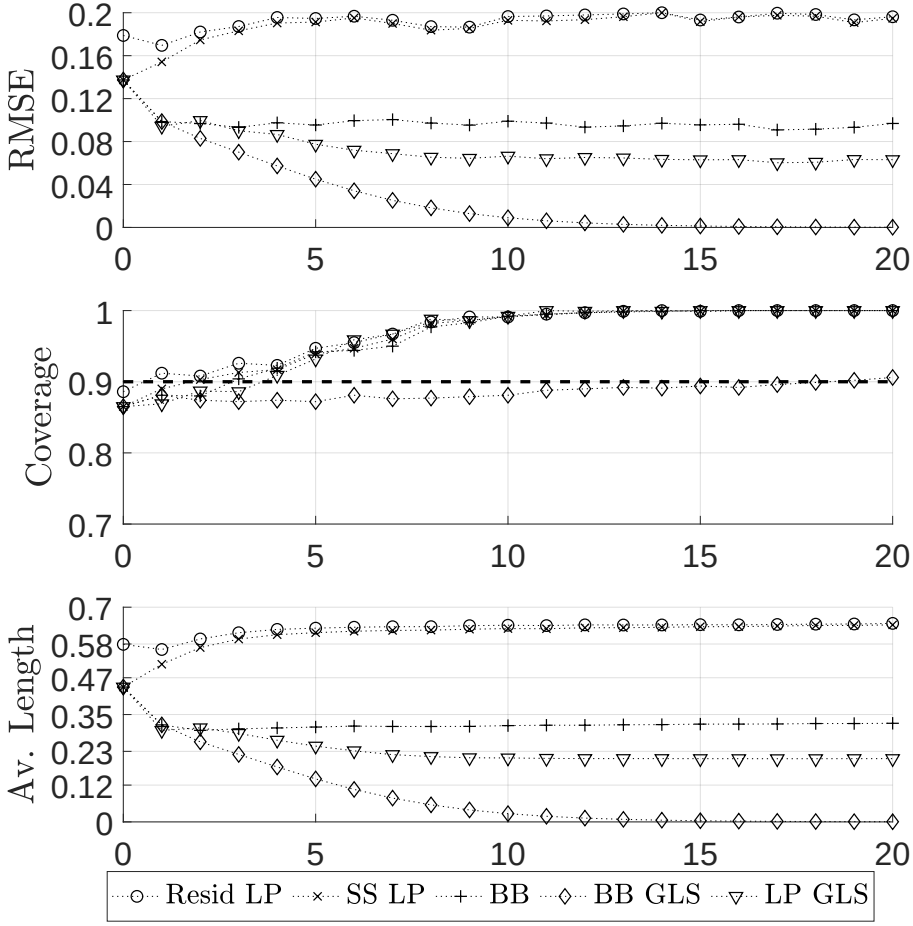




Finding 3: $\widehat{\Theta}_{LP}^{resid}$ and $\widehat{\Theta}_{LP}^{ss}$ not competitive

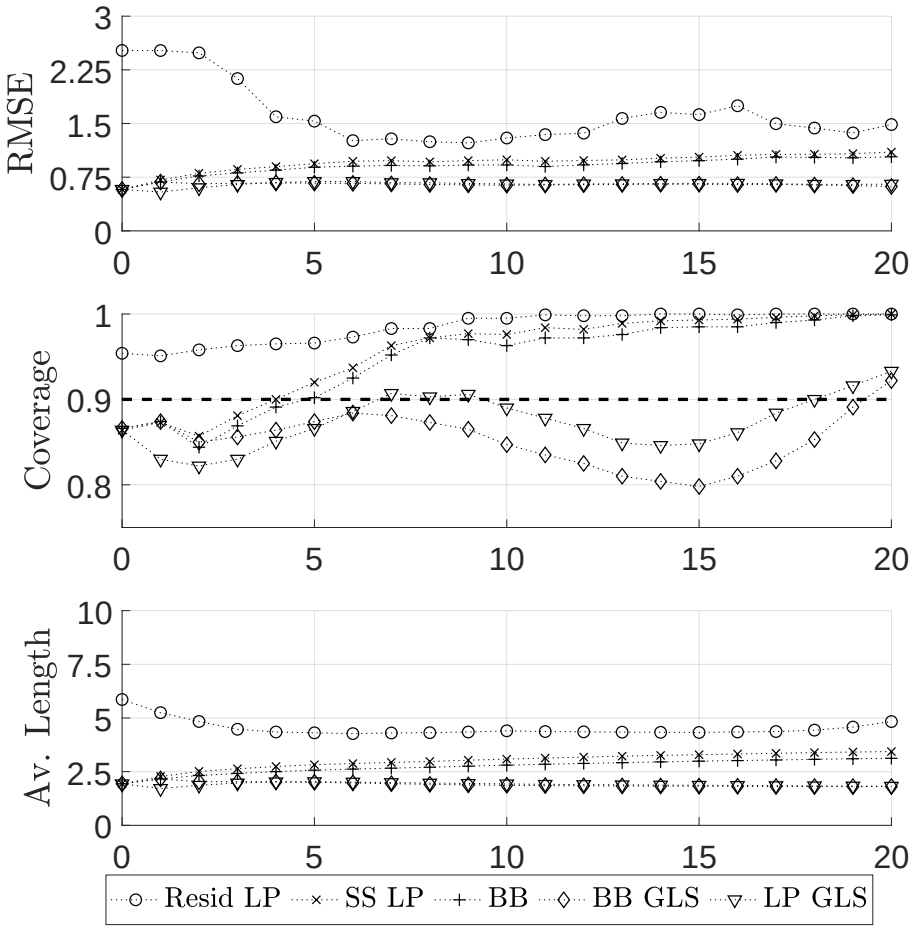
Pointwise MBB 90% confidence bands, $a_{11} = 0.5$, $T = 500$, $p = 1$, with bias corr., *corr.* $= 0.9$ (left), *corr.* $= 0.5$ (right), DGP1

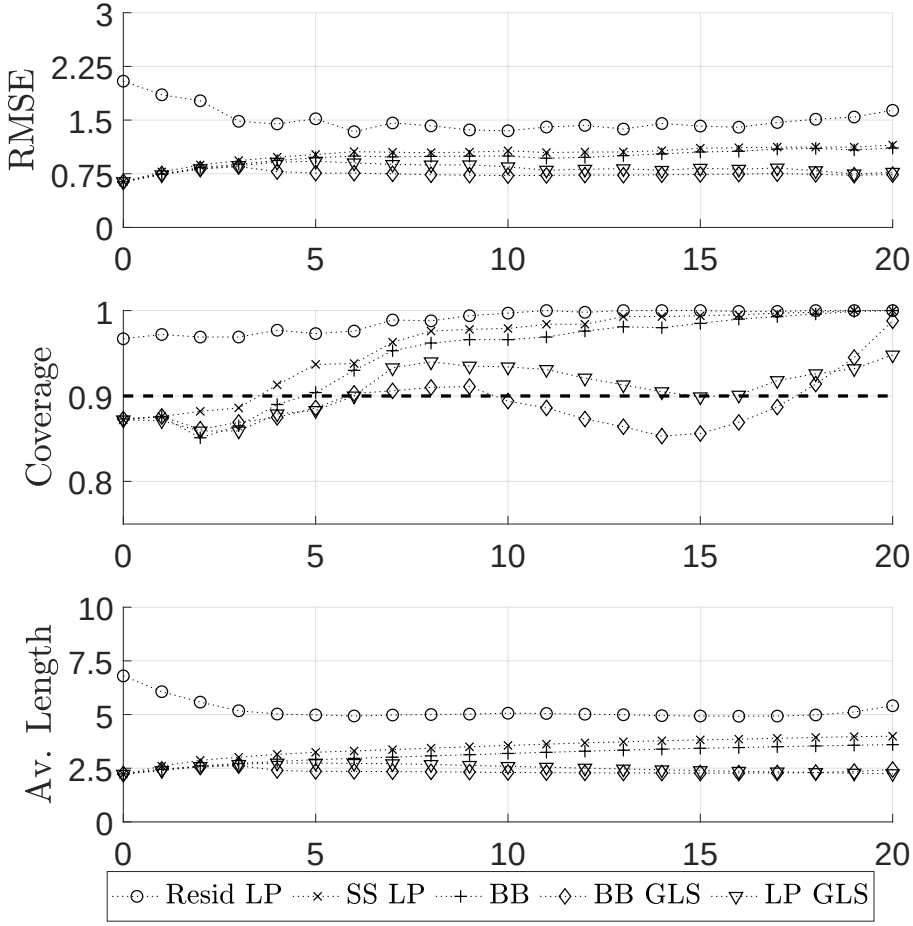




Finding 3: $\widehat{\Theta}_{LP}^{resid}$ and $\widehat{\Theta}_{LP}^{ss}$ not competitive

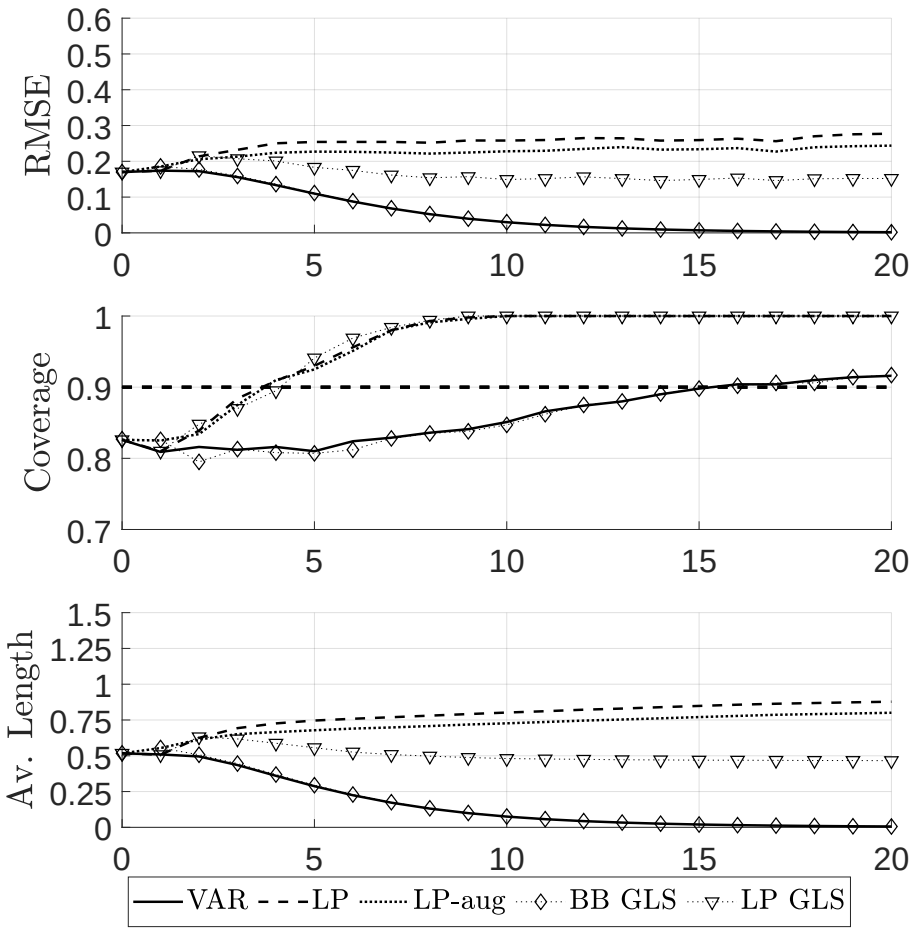
Pointwise MBB 90% confidence bands, $T = 200$, *corr.* $= 0.9$, with bias corr., $p = 1$ (left), $p = 4$ (right), DGP2

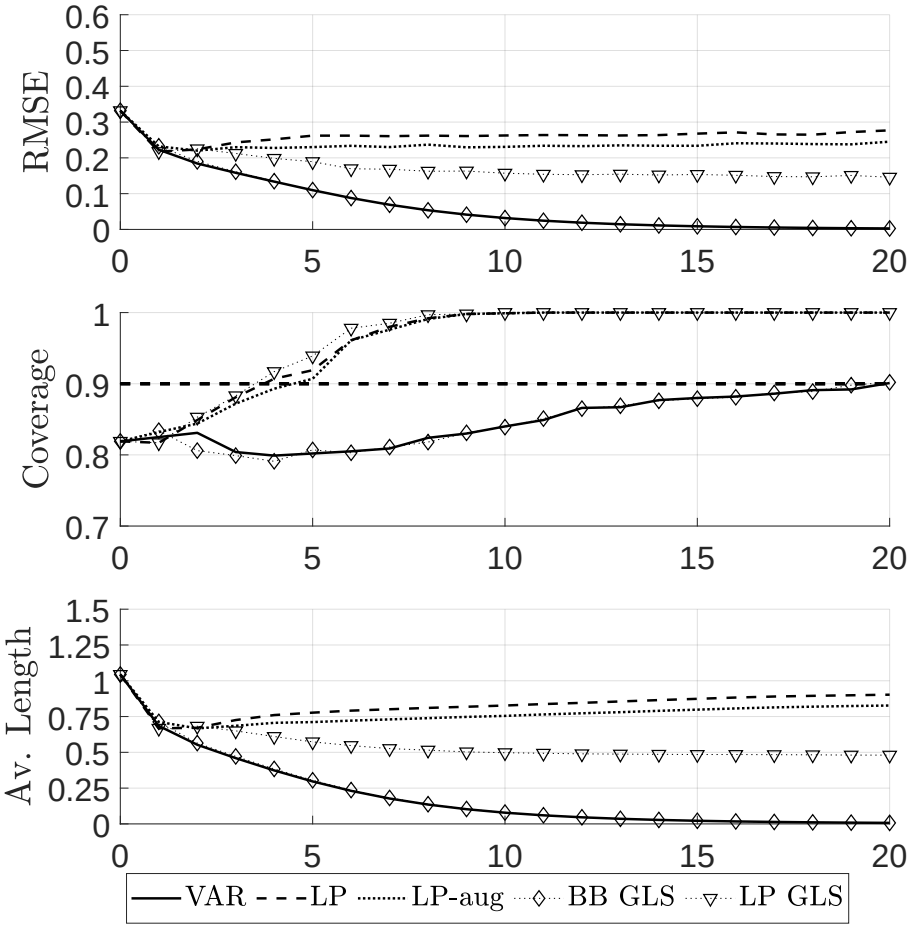




Finding 4: $\widehat{\Theta}_{BB}^{GLS}$ dominates overall, $\widehat{\Theta}_{LP}^{GLS}$ is competitive

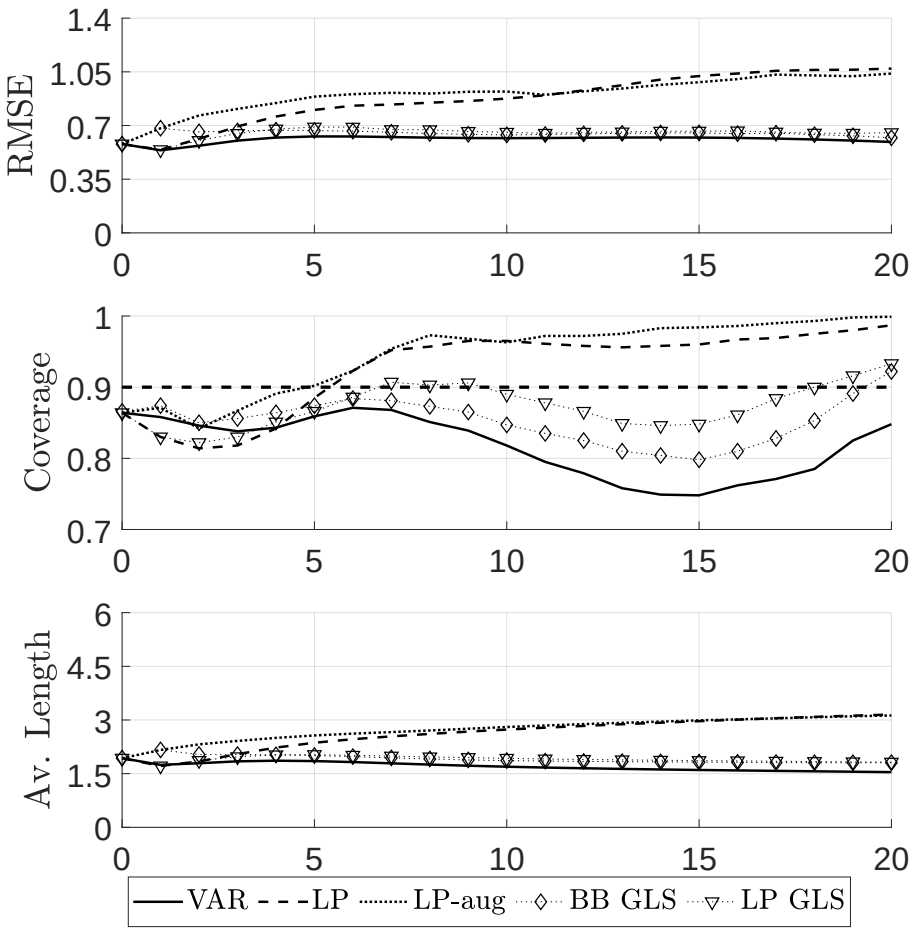
Pointwise MBB 90% confidence bands, $a_{11} = 0.5$, $T = 100$, $p = 1$, with bias corr., *corr.* $= 0.9$ (left), *corr.* $= 0.5$ (right), DGP1

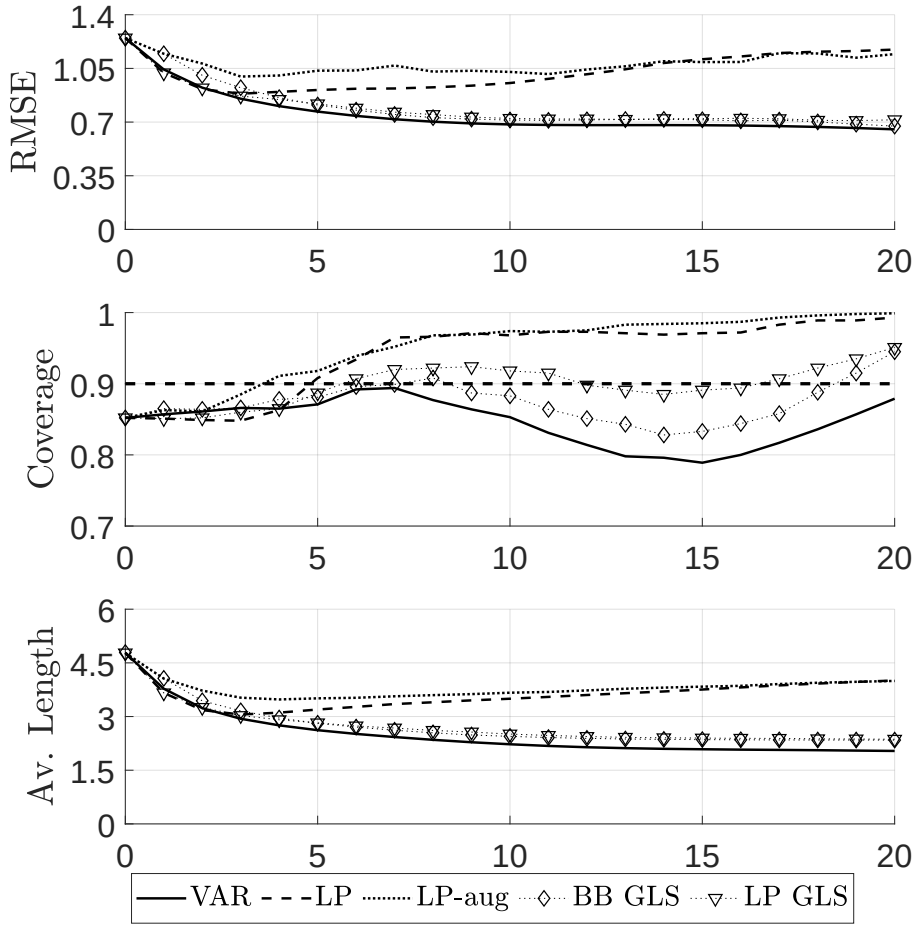




Finding 4: $\widehat{\Theta}_{BB}^{GLS}$ dominates overall, $\widehat{\Theta}_{LP}^{GLS}$ is competitive

Pointwise MBB 90% confidence bands, $T = 200$, $p = 1$, with bias corr.
corr. $= 0.9$ (left), *corr.* $= 0.5$ (right), DGP2





Conclusion

- Comparison of 10 LP estimators to VAR estimator
- Numerical equivalences and a Monte Carlo study
- 4 findings:
  1. Numerical Equivalence between $\widehat{\Theta}_{LP}^{aug}$ and $\widehat{\Theta}_{BB}$
  2. $\widehat{\Theta}_{LP}$ and $\widehat{\Theta}_{LP}^{aug}$ dominate $\widehat{\Theta}_{IV}$,$\widehat{\Theta}_{IV}^{y}$, and $\widehat{\Theta}_{IV}^{yz}$
  3. $\widehat{\Theta}_{LP}^{resid}$ and $\widehat{\Theta}_{LP}^{ss}$ not competitive
  4. $\widehat{\Theta}_{BB}^{GLS}$ dominates overall, $\widehat{\Theta}_{LP}^{GLS}$ is competitive

Summary

- VAR and LP are two alternative ways of estimating IRFs
- They have the same estimand
- But in small sample their bias-variance trade-off differs
- Illustrative example
- Comparison of different proxy LP estimators

Breitung, J. & Brüggemann, R. (2023), 'Projection estimators for structural impulse responses', *Oxford Bulletin of Economics and Statistics* **85**, 1320–1340.

Bruns, M. & Lütkepohl, H. (2022), 'Comparison of local projection estimators for proxy vector autoregressions', *Journal of Economic Dynamics & Control* **134**, 104277.

Jentsch, C. & Lunsford, K. G. (2019), 'The dynamic effects of personal and corporate income tax changes in the United States: Comment', *American Economic Review* **109**, 2655–2678.

Jordà, O. (2005), 'Estimation and inference of impulse responses by local projection', *American Economic Review* **95**, 161–182.

Kilian, L. & Lütkepohl, H. (2017), *Structural Vector Autoregressive Analysis*, Cambridge University Press, Cambridge.

Li, D., Plagborg-Møller, M. & Wolf, C. K. (2024), 'Local projections vs. VARs: Lessons from thousands of DGPs', *Journal of Econometrics* **244**, 105722.

Lusompa, A. (2023), 'Local projections, autocorrelation, and efficiency', *Quantitative Economics* **14**, 1199–1220.

Mertens, K. & Ravn, M. O. (2013), 'The dynamic effects of personal and corporate income tax changes in the United States', *American Economic Review* **103**, 1212–1247.

Montiel Olea, J. L. & Plagborg-Møller (2021), 'Local projection inference is simpler and more robust than you think', *Econometrica* **89**, 1789–1823.

Montiel Olea, J. L., Plagborg-Møller, M., Qian, E. & Wolf, C. (2025), Local projections or VARs? A primer for macroeconomists, Technical report, NBER Working Paper.

Plagborg-Møller, M. & Wolf, C. K. (2021), 'Local projections and VARs estimate the same impulse responses', *Econometrica* **89**, 955 – 980.

Stock, J. H. & Watson, M. W. (2018), 'Identification and estimation of dynamic causal effects in macroeconomics using external instruments', *Economic Journal* **128**, 917–948.

VAR

1 VAR estimator (Kilian & Lütkepohl 2017)

$$y_t = \nu + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t, \quad u_t \sim (0, \Sigma_u) \tag{1}$$

$$\hat{\theta}_0 = \frac{\sum_{t=1}^{T} (\hat{u}_t z_t)}{\sum_{t=1}^{T} \hat{u}_{1t} z_t} \tag{4}$$

$$\hat{\Theta}(H)_{VAR} = [\hat{\theta}_0, \hat{\Phi}_1 \hat{\theta}_0, ..., \hat{\Phi}_H \hat{\theta}_0],$$

where

- $\hat{\Phi}_h = \sum_{j=1}^{i} \Phi_{i-j} A_j$
- A_i estimated equation-by-equation OLS of (5) (and potentially bias-corrected)
- $\hat{\theta}_0 = \hat{b}$ from equation (8)

back

LP

2 LP estimator (Jordà 2005)

$$y_{t+h} = \nu_h + A^{(h+1)} Y_{t-1} + v_{t+h}^{(h)}, \tag{2}$$

$$\hat{\Theta}(H)_{LP} = [\hat{\theta}_0, \hat{\Phi}_1^{LP}\hat{\theta}_0, ..., \hat{\Phi}_H^{LP}\hat{\theta}_0], \tag{11}$$

where

- $\hat{\theta}_0 = \hat{b}$ from equation (8)
- $\hat{\Phi}_i^{LP} = \hat{A}_i^{(i)}$ from equation (6)

back

LP GLS

3 LP GLS estimator (Lusompa 2023)

$$\tilde{y}_{t+h} = \nu_h + A^{(h+1)} Y_{t-1} + e_t^{(h)}, \tag{12}$$

where $\tilde{y}_{t+h} = y_{t+h} - \hat{\Phi}_1^{GLS} \hat{u}_{t+h-1} - \cdots - \hat{\Phi}_{h-1}^{GLS} \hat{u}_{t+1}$ and $\hat{\Phi}_1^{GLS}, \ldots, \hat{\Phi}_{h-1}^{GLS}$ are obtained from the regressions at horizons 1 through $h-1$ as the first K columns of the estimator $\hat{A}^{(i)}$.

$$\widehat{\Theta}_{LP}^{GLS} = [\hat{\theta}_0, \hat{\Phi}_1^{LP} \hat{\theta}_0, \hat{\Phi}_2^{GLS} \hat{\theta}_0, \ldots, \hat{\Phi}_H^{GLS} \hat{\theta}_0], \tag{13}$$

back

LP GLS

4 Lag-augmented LP estimator (Montiel Olea & Plagborg-Møller 2021)

$$\hat{\Theta}(H)^{aug}_{LP} = [\hat{\theta}_0, \hat{\Phi}^{aug}_1 \hat{\theta}_0, ..., \hat{\Phi}^{aug}_H \hat{\theta}_0], \tag{14}$$

where

- $\hat{\theta}_0 = \hat{b}$ from equation (8)
- $\hat{\Phi}^{aug}_i = \hat{A}^{(i)}_i$ from equation (plus one lag) (6)

back

BB

5 OLS estimator by Breitung & Brüggemann (2023)

$$\hat{\Theta}(H)^{ols}_{BB} = [\hat{\theta}^{ols}_0, \hat{\theta}^{ols}_1, ..., \hat{\theta}^{ols}_H], \tag{15}$$

- from OLS regression: $y_{t+h} = \nu_h + A^{(h+1)} Y_{t-1} + \theta_h \hat{w}_{1t} + e^{(h)}_t$
- where $\hat{w}_{1t}$ obtained in a preliminary projection step:
  a. Estimate $(\hat{u}_{2t}, .., \hat{u}_{Kt}) = \gamma \hat{u}_{1t} + w^{(2)}_t$ via instrumental variables estimator
  b. Estimate $\hat{u}_{1t} = \eta \hat{w}^{(2)}_t + w_{1t}$

back

BB GLS

6 GLS estimator by Breitung & Brüggemann (2023)

$$\hat{\Theta}(H)^{gls}_{BB} = [\hat{\theta}^{ols}_0, \hat{\theta}^{ols}_1, \hat{\theta}^{gls}_1, ..., \hat{\theta}^{gls}_H], \tag{16}$$

- from OLS regression:
  $y_{t+h} = \nu_h + A^{(h+1)} Y_{t-1} + \Phi_1 \hat{u}_{t+h-1} + ... + \Phi_{h-2} \hat{u}_{t+2} + \theta_h \hat{w}_{1t} + e^{(h)}_t$
- where $\hat{w}_{1t}$ obtained is obtained as above

back

IV

7 Instrumental Variable estimator (Stock & Watson 2018)

$$\hat{\Theta}(H)_{IV} = [\hat{\theta}_0^{iv}, \hat{\theta}_1^{iv}, ..., \hat{\theta}_H^{iv}], \tag{17}$$

- From IV regression: $y_{t+h} = x_t + \theta_h y_{1t} + u_{t+h}^{(h)}$
- $x_t = 1$
- Using z_t as an instrument for y_{1t}

back

IV^y

8 Instrumental Variable estimator (Stock & Watson 2018)

$$\hat{\Theta}(H)^y_{IV} = [\hat{\theta}^{iv}_{0,y}, \hat{\theta}^{iv}_{1,y}, ..., \hat{\theta}^{iv}_{H,y}], \tag{18}$$

- From IV regression: $y_{t+h} = x_t + \theta_h y_{1t} + u^{(h)}_{t+h}$
- $x_t = (1, y'_{t-1}, \ldots, y'_{t-p})'$
- Using z_t as an instrument for y_{1t}

back

IV^{yz}

9 Instrumental Variable estimator (Stock & Watson 2018)

$$\hat{\Theta}(H)_{IV}^{yz} = [\hat{\theta}_{0,yz}^{iv}, \hat{\theta}_{1,yz}^{iv}, ..., \hat{\theta}_{H,yz}^{iv}], \tag{19}$$

- From IV regression: $y_{t+h} = x_t + \theta_h y_{1t} + u_{t+h}^{(h)}$
- $x_t = (1, y_{t-1}', \ldots, y_{t-p}', z_{t+1}, \ldots, z_{t+h})'$
- Using z_t as an instrument for y_{1t}

back

LP resid

10 Residual-based LP-type estimator

$$\hat{\Theta}(H)_{LP}^{resid} = [\hat{\theta}_0^{LPresid}, \hat{\theta}_1^{LPresid}, ..., \hat{\theta}_H^{LPresid}], \tag{20}$$

"LP representation"

$$y_{t+h} = \nu_h + A^{(h+1)} Y_{t-1} + v_{t+h}^{(h)} \qquad (2) \tag{21}$$

- Noting that $\mathbb{E}(v_{t+h}^{(H)} z_{t+H-h}) = \Phi_h \theta_0$
- $\hat{\theta}_h^{LPresid} = \frac{1}{T} \sum_{t=1}^{T} \hat{v}_{t+h}^{(H)} z_{t+h}$
- $\hat{v}_{t+H}^{(H)}$ obtained via OLS regression of equation (6) at horizon H

back

LP ss

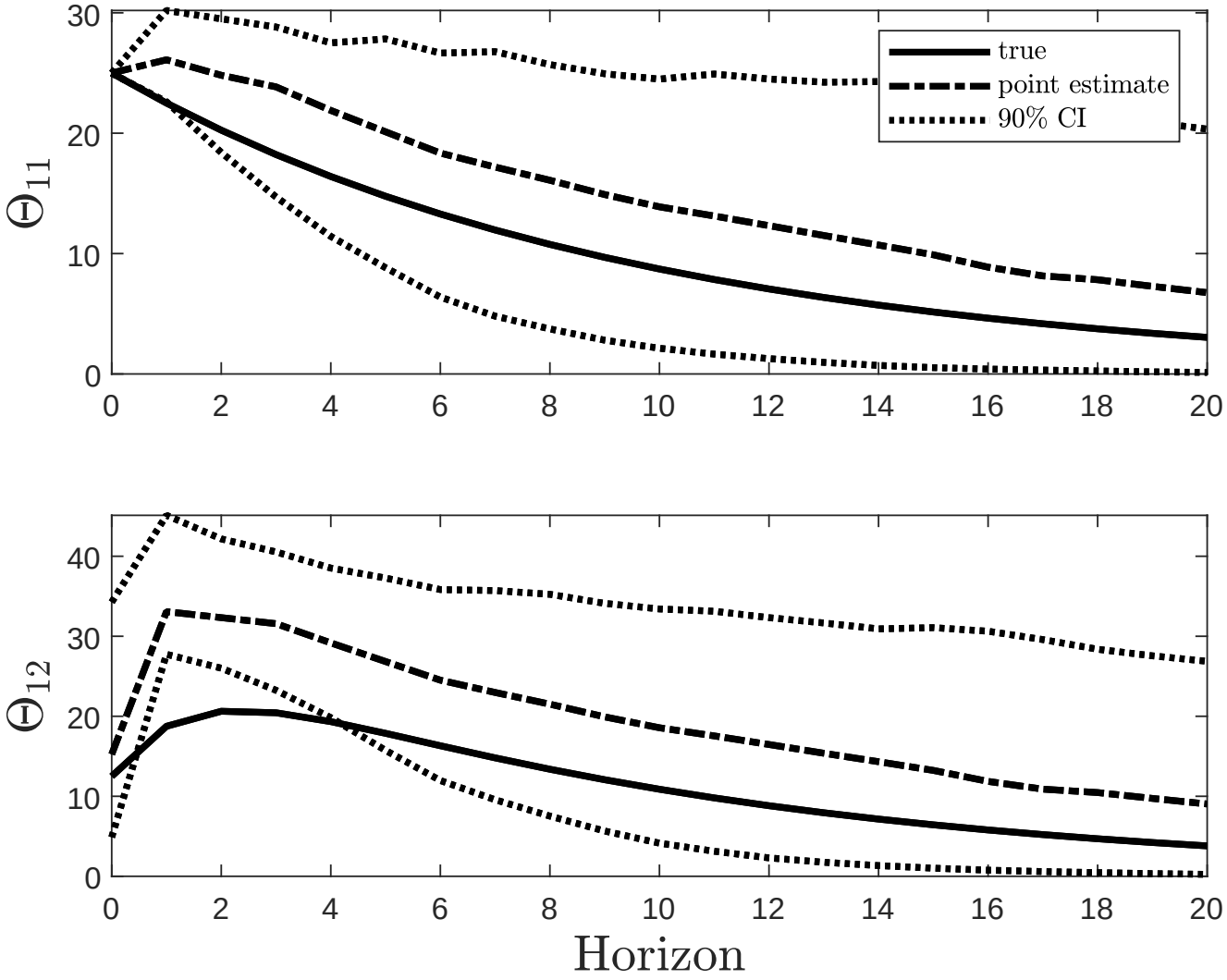
11 Residual-based LP-type estimator (small-scale)

$$\hat{\Theta}(H)^{ss}_{LP} = [\hat{\theta}_0, \hat{\theta}^{LPss}_1, ..., \hat{\theta}^{LPss}_H], \tag{22}$$

- Estimate all $H+1$ equations $y_{t+h} = \nu_h + A^{(h+1)} Y_{t-1} + v^{(h)}_{t+h}$
- Compute $\hat{\theta}^{LPss}_h = \frac{1}{T} \sum_{t=1}^{T} \hat{v}^{(h)}_{t+h} z_{t+h}$

back

Coverage, Average Length, and Mean Squared Error



Coverage, Average Length, and Mean Squared Error

- Consider N generated data sets
- Confidence interval: $\{CI_l(h), CI_h(h)\}$
- Average Length: $AL(h) = \frac{1}{N}\sum_{n=1}^{N}(CI_l(h) - CI_h(h))$
- Coverage: $Cov(h) = \frac{1}{N}\sum_{n=1}^{N}\mathbb{1}\{CI_l(h) < \Theta_h^{true} < CI_h(h)\}$
- Mean Squared Error: $MSE(h) = \frac{1}{N}\sum_{n=1}^{N}(\hat{\Theta}_h - \Theta_h^{true})^2$

Moving Block Bootstrap and Bias Correction

- Quantitatively very relevant: bootstrap algorithm
- We use Moving Block Bootstrap as in Jentsch & Lunsford (2019)

  ▸ MBB
- OLS estimated VAR coefficients are biased in small sample
- We employ the bias-correction in **?** for all bootstrap-residuals

  ▸ Bias Correction
- In addition, the estimators BB, BB-GLS, LP-SS are computed using bias-corrected VAR residuals

Moving Block Bootstrap

1. A VAR(p) model is fitted to the sample by LS and bias-adjusted, giving bootstrap estimates $\hat{A}^{(n)}$,

$$\hat{\Phi}_i^{(n)} = \sum_{j=1}^{i} \hat{\Phi}_{i-j}^{(n)} \hat{A}_j^{(n)}, \quad i = 1, \ldots, H, \quad \text{with } \hat{\Phi}_0^{(n)} = I_K,$$

   and residuals $\hat{u}_t^{(n)}$.

2. Then bootstrap estimates

$$\hat{b}^{(n)} = \sum_{t=1}^{T} \hat{u}_t^{(n)} z_t^{(n)} \Big/ \sum_{t=1}^{T} \hat{u}_{1t}^{(n)} z_t^{(n)} .$$

   of the structural parameters are determined.

3. Finally bootstrap estimates of the impulse responses of interest are computed as

$$\widehat{\Theta}(H)^{(n)} = [\hat{b}^{(n)}, \hat{\Phi}_1^{(n)} \hat{b}^{(n)}, \ldots, \hat{\Phi}_H^{(n)} \hat{b}^{(n)}]$$

   and stored.

Moving Block Bootstrap

A block length $\ell < T$ has to be chosen for the MBB. The blocks of length ℓ of the estimated residuals and proxies are arranged in the form of the matrix

$$\begin{bmatrix} \begin{pmatrix} \hat{u}_1 \\ z_1 \end{pmatrix} & \begin{pmatrix} \hat{u}_2 \\ z_2 \end{pmatrix} & \cdots & \begin{pmatrix} \hat{u}_\ell \\ z_\ell \end{pmatrix} \\ \begin{pmatrix} \hat{u}_2 \\ z_2 \end{pmatrix} & \begin{pmatrix} \hat{u}_3 \\ z_3 \end{pmatrix} & \cdots & \begin{pmatrix} \hat{u}_{1+\ell} \\ z_{1+\ell} \end{pmatrix} \\ \vdots & \vdots & & \vdots \\ \begin{pmatrix} \hat{u}_{T-\ell+1} \\ z_{T-\ell+1} \end{pmatrix} & \begin{pmatrix} \hat{u}_{T-\ell+2} \\ z_{T-\ell+2} \end{pmatrix} & \cdots & \begin{pmatrix} \hat{u}_T \\ z_T \end{pmatrix} \end{bmatrix}.$$

$$\tilde{u}_{j\ell+i} = \hat{u}_{j\ell+i} - \frac{1}{T-\ell+1} \sum_{r=0}^{T-\ell} \hat{u}_{i+r}$$

$$\tilde{z}_{j\ell+i} = z_{j\ell+i} - \frac{1}{T-\ell+1} \sum_{r=0}^{T-\ell} z_{i+r}$$

$$\begin{pmatrix} u_t^{MBB} \\ z_t^{MBB} \end{pmatrix}, \quad t = 1, \ldots, T.$$

Bias Correction (?)

$$\widehat{\mathbf{A}}^c = \widehat{\mathbf{A}} + \frac{1}{T}\widehat{G}\Big[(I - \widehat{\mathbf{A}}')^{-1} + \widehat{\mathbf{A}}'(I - \widehat{\mathbf{A}}'^2)^{-1} + \sum_{\lambda \in Spec(\widehat{\mathbf{A}})} \lambda(I - \lambda\widehat{\mathbf{A}}')^{-1}\Big]\widehat{\Gamma}(0)^{-1},$$

where $\widehat{\Gamma}(0) = T^{-1}\sum_{t=1}^{T}(Y_t - \bar{y})(Y_t - \bar{y})'$, $\widehat{G} = T^{-1}\sum_{t=1}^{T}\hat{U}_t\hat{U}_t'$ and $Spec(\mathbf{A})$ denotes the set of eigenvalues λ of the $\mathbf{A}$ matrix.

Companion Matrix

$$\mathbf{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_{p-1} & A_p \\ I_K & 0 & \dots & 0 & 0 \\ 0 & I_K & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_K & 0 \end{bmatrix}$$

Numerical Equivalences

Table: Equality of Estimators Based on OLS Estimation

	$\widehat{\Theta}_{VAR}$	$\widehat{\Theta}_{LP}$	$\widehat{\Theta}^{GLS}_{LP}$	$\widehat{\Theta}^{aug}_{LP}$	$\widehat{\Theta}_{BB}$	$\widehat{\Theta}^{GLS}_{BB}$	$\widehat{\Theta}_{IV}$	$\widehat{\Theta}^{y}_{IV}$	$\widehat{\Theta}^{yz}_{IV}$	$\widehat{\Theta}^{resid}_{LP}$	$\widehat{\Theta}^{ss}_{LP}$
$\widehat{\Theta}_{VAR}$	all h										
$\widehat{\Theta}_{LP}$	$h = 0, 1$	all h									
$\widehat{\Theta}^{GLS}_{LP}$	$h = 0, 1$	$h = 0, 1$	all h								
$\widehat{\Theta}^{aug}_{LP}$	$h = 0$	$h = 0$	$h = 0$	all h							
$\widehat{\Theta}_{BB}$	$h = 0$	$h = 0$	$h = 0$	all h	all h						
$\widehat{\Theta}^{GLS}_{BB}$	$h = 0$	$h = 0$	$h = 0$	$h = 0, 1$	$h = 0, 1$	all h					
$\widehat{\Theta}_{IV}$	–	–	–	–	–	–	all h				
$\widehat{\Theta}^{y}_{IV}$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	–	all h			
$\widehat{\Theta}^{yz}_{IV}$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	–	$h = 0$	all h		
$\widehat{\Theta}^{resid}_{LP}$	–	–	–	–	–	–	–	–	–	all h	
$\widehat{\Theta}^{ss}_{LP}$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	$h = 0$	–	$h = 0$	$h = 0$	–	all h

Note: h denotes the propagation horizon.

back